Some analysis on generic p-Laplacian boundary value problems of weighted fractional impulsive differential equations

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This paper's focus to deals with some analysis results for generic p-Laplacian boundary value problems of weighted fractional impulsive differential equations via of some celebrated fixed theorems, new results are given.

In this present paper, we are interested in the existence and uniqueness results on generic p-Laplacian boundary value problems of weighted fractional impulsive differential equations:

$$\begin{cases} {}^{\psi_2;C}_{\ \ t}\mathbf{D}^{\beta}_{T^-}\left(\rho(t)\phi_p\left({}^{\psi_1;C}_{\ \ t}\mathbf{D}^{\alpha}_{a^+}u\right)\right)(t)+s(t)g\left(u(t)\right)=f(t,u(t)),\ a<t<T,\\ u(a)=u_0, \Delta\left(u\left(t_k\right)\right)=I^1_k\left(u\left(t_k\right)\right), k=1,2,\ldots,m,\\ {}^{\psi_1;C}\mathbf{D}^{\alpha}_{a^+}u(T)=u_1, \Delta\phi_p\left({}^{\psi_1;C}_{\ \ t}\mathbf{D}^{\alpha}_{a^+}u\right)(t_k)=I^2_k\left(u\left(t_k\right)\right), \end{cases} \tag{1}$$

where ${}^{\psi_2;C}_{\ \ t}\mathbf{D}^{\beta}_{T^-}$, ${}^{\psi_1;C}_{\ \ t}\mathbf{D}^{\alpha}_{a^+}$ are the left and right-sided ψ-Caputo fractional derivatives with with respect to ψ_1 and $p>1$, $0<\alpha,\beta\leqq 1, \phi_p$ is a p -Laplacian operator, $s(t),\rho(t)\in C([a,T],\mathbb{R}^{\star}_{+})$, $f,g\in C([a,T]\times\mathbb{R},\mathbb{R}),\quad u_0,u_1,\lambda\in\mathbb{R}$, for $k=1,2,\ldots,m,\quad I^i_k\in C(\mathbb{R},\mathbb{R}), i=1,2,\ ,a=t_0<t_1<\cdots<t_k<\cdots<t_m<t_{m+1}=T$. $\Delta u\left(t_k\right)=u\left(t_k^+\right)-u\left(t_k^-\right), u\left(t_k^+\right)$ and $u\left(t_k^-\right)$ denote the right and the left limits of $u(t)$ at $t=t_k$ $(k=1,2,\ldots,m)$, respectively, and $\Delta\phi_p\left({}^{\psi_1;C}_{\ \ t}\mathcal{D}^{\alpha}_{a^+}u\right)(t_k)$ has a similar meaning for $\phi_p\left({}^{\psi_1;C}_{\ \ t}\mathcal{D}^{\alpha}_{a^+}u\right)(t_k)$.

Definition 1 (PC-Function space)**.** Let $PC(J,\mathbb{R})=\{u:J\to\mathbb{R}:u\in C\left(J_k,\mathbb{R}\right)$ for $k=1,2,\ldots,m$ and there exist $u\left(t_k^+\right)$ and $u\left(t_k^-\right)$ at $t=t_k$ with $u\left(t_k^-\right)=u\left(t_k\right)\}$. Then $PC(J,\mathbb{R})$ is a Banach space endowed with the norm $\|u\|=\sup|u(t)|$.

Definition 2. Let ψ be strictly increasing and n times differentiable function on J, then $AC^n_{\psi}(\ J)=\left\{u:\ J\to\mathbb{R}\quad\text{and } \delta^{[n-1]}_{\psi}u\in AC(\ J),\ \ \delta^{[n-1]}_{\psi}u=\left(\frac{1}{\psi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)u\right\}$ denotes the Banch space of n times absolutely continuous with respect to the strictly increasing differentiable function ψ.

Definition 3 (Heavside function H)**.** We define the Heavside function as follows

$$H(t)=\begin{cases}1 & \text{if } t\geqq 0\\ 0 & \text{else.}\end{cases}$$

Proposition 1. *Let u be a piecewise function ($u\in PC(J,\mathbb{R})$), $t_1,t_2,\cdots,t_k$ for $k=1,2,\cdots,m$ the fixed moments of impulsive effect and $\varrho^k=u(t_k^+)-u(t_k^-)$ the magnitude and direction of the impulsive effect at t_k. Then u can be written as the some of a continuous function g and the Heavside functions*

$$u(t)=g(t)+\sum_{j=0}^{k}\varrho^j H(t-t_j),$$

where $\varrho^0=0$.

Lemma 1. *Assume that $f, g \in C(J \times \mathbb{R}, \mathbb{R})$, then $u(t) \in PC(J, \mathbb{R})$ is a solution of the boundary value problem (1) if and only if $u(t)$ is a solution of the integral equation*

$$u(t) = \begin{cases} u_0 + \frac{1}{\Gamma(\alpha)} \int_a^t \psi_1'(s)(\psi_1(t) - \psi_1(s))^{\alpha-1} \phi_{p^\star}(\mathfrak{F}\mathcal{N}u(s))\mathrm{d}s \\ + \sum_{j=0}^k I_j^1 \left(u\left(t_j\right)\right) H(t - t_j) \quad t \in J, k = 0, 1, \ldots, m, \end{cases}$$

where

$$\mathfrak{F}\mathcal{N}u(t) = \frac{1}{\rho(t)} \begin{cases} \rho(T)\phi_p\left(u_1\right) + \frac{1}{\Gamma(\beta)} \int_t^T \psi_2'(\tau)(\psi_2(\tau) - \psi_2(t))^{\beta-1} \mathcal{N}u(\tau)\mathrm{d}\tau \\ + \sum_{j=0}^k I_j^2(u(t_j)H(t - t_j), \ t \in J, k = 1, \ldots, m \end{cases}$$

and $\mathcal{N}$ is the Nemytskii operator associated to (1) definded by this idendification

$$\mathcal{N}(u(t)) = f(t, u(t)) - s(t)g\left(u(t)\right), \quad t \in J, u \in PC(J, \mathbb{R}).$$

Theorem 1. *Suppose that there exists positive constants $\Lambda_1,\ \Lambda_2,\ \Lambda_3$, and Λ_4 such that*

$$\Lambda_1 \leqq \mid \mathfrak{F}\mathcal{N}u(t) \mid \leqq \Lambda_2;$$

$$\Lambda_3 \leqq \parallel u \parallel \leqq \Lambda_4;$$

for $\forall\ t \in J, u \in PC(J, \mathbb{R})$. If suitable conditions hold, then the problem (1) has a unique solution.

In the survey paper, we present some background material for our problems and by applying Schauder's, Schaefer's fixed point theorems, and Banach contraction mapping principle, we will prove the existence and uniqueness of solutions for the problem (1).

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