ON SOME PROPERTIES OF MEROMORPHIC SOLUTIONS OF Q-DIFFERENCE EQUATIONS IN ULTRAMETRIC FIELDS

Salih Bouternikh[1], Houda Boughaba[2], Tahar Zerzaihi[3]

[1]Laboratory of Pure and Applied Mathematics (LMPA), University of Mohamed Seddik Ben Yahia, Jijel, Algeria

[2]Laboratory of Pure and Applied Mathematics (LMPA), University of Mohamed Seddik Ben Yahia, Jijel, Algeria

[3]Laboratory of Pure and Applied Mathematics (LMPA), University of Mohamed Seddik Ben Yahia, Jijel, Algeria

bouternikhsalah18@gmail.com, houdaboughaba5@gmail.com, zerzaihi@yahoo.com

Let $\mathbb{K}$ be an ultrametric algebraically closed field of characteristic zero, complete for an ultrametric absolute value $|\cdot|$.

Recently, many papers (including [2, 3, 4]) focused on differences and q-difference equations in both complex and ultrametric fields. The authors obtained many meaningful results about the growth order of their solutions. In this paper, we will generalize some of these results to the case of ultrametric q-difference equations.

In what follows, we denote by $\mathcal{A}(\mathbb{K})$ the $\mathbb{K}$-algebra of entire function in $\mathbb{K}$, i.e, the set of power series with an infinite radius of convergence, $\mathcal{M}(\mathbb{K})$ the field of meromorphic function in $\mathbb{K}$, i.e, the field of fractions of $\mathcal{A}(\mathbb{K})$.

A meromorphic function (resp. an entire) is said to be transcendental if it is not a rational function (resp. not a polynomial).

Let $f(x) = \sum_{n\geq 0} a_n x^n$ be a ultrametric entire function. For all $r > 0$, we denote by $|f|(r) = \max\limits_{n\geq 0}|a_n|r^n$, the maximum modulus of f. This is extended to meromorphic functions $h = f/g$ by $|h|(r) = |f|(r)/|g|(r)$. Throughout this paper, we use the ultrametric Nevanlinna theory, so we have to recall some basic notions of this theory (see [1]).

For every $r \in]0, R[$, let f be a non-constant meromorphic function on the disk $d(0, r)$. Using the notation $\log^+ a = \max\{\log a, 0\}$ (where log is the real logarithm function), we define the compensation function by

$$m(r, f) = \log^+ |f|(r) = \max\{\log |f|(r), 0\}.$$

Let $f \in \mathcal{M}(d(0, R^-))$ such that 0 is neither a zero nor a pole of f. For every $r \in]0, R[$, we denote by $Z(r, f)$ the counting function of zeros of f in the disk $d(0, r)$ counting multiplicity and $N(r, f)$ the counting function of poles of f in the disk $d(0, r)$ counting multiplicity. They are defined by

$$Z(r, f) = \sum_{\omega_\alpha(f)>0, |\alpha|\leq r} \omega_\alpha(f) \log \frac{r}{|\alpha|}$$

and

$$N(r, f) = Z\left(r, 1/f\right) = -\sum_{\omega_\alpha(f)<0, |\alpha|\leq r} \omega_\alpha(f) \log \frac{r}{|\alpha|}.$$

Finally, we set $T(r, f) = N(r, f) + m(r, f)$. The function $r \mapsto T(r, f)$ is called the Nevanlinna function or characteristic function of Nevanlinna. Moreover, Similarly to definition known on complex case, given $f \in \mathcal{M}(\mathbb{K})$, the superior limit

$$\rho(f) = \limsup_{r\to+\infty} \frac{\log T(r, f)}{\log r} \tag{1}$$

is called the order of growth of f or the order of f.
In this part, we consider ultrametric q-difference equations of the form

$$\big(f(qx)+f(x)\big)\big(f(x)+f\big(\frac{x}{q}\big)\big)=\frac{P(x,f(x))}{Q(x,f(x))}, \tag{1}$$

$$\sum_{i=1}^{n} A_j(x)f(q^j x)=\frac{P(x,f(x))}{Q(x,f(x))}, \tag{2}$$

where $P(x,f(x))=\sum_{i=0}^{p} a_i(x)f(x)^i$ and $Q(x,f(x))=\sum_{i=0}^{d} b_i(x)f(x)^i$ are relatively prime polynomials in f and $a_i(x)$ for $i=0,\ldots,p,$ $b_i(x)$ for $i=0,\ldots,d$ are polynomials with $a_p(x)b_d(x)\neq 0$, $A_1(x),\cdots,A_n(x), a_j(x),$ $b_j(x)$ are rational functions and $q\in\mathbb{K}$.

In the following, we denote by $p=\deg_f P$ *and* $d=\deg_f Q$, the degree of the numerator P and the denominator Q, respectively. In the same way, we suppose $m=\max\{p,d\}>4$ *and* $p-d\geq 3$.
We obtained the following results.

Theorem 1. *If $q\in\mathbb{K}$ satisfies $|q|=1$, the equation (1) has no transcendantal entire solution neither transcendantal meromorphic solution which has finitely many poles.*

Theorem 2. *Let f be a transcendantal meromorphic solution equation (1) and $q\in\mathbb{K}$ satisfies $|q|>1$. Then, we have*

$$\rho(f)\geq\frac{\log m-\log 4}{\big|\log|q|\big|}.$$

Theorem 3. *Let f be a transcendental meromorphic solution of equation (2) and $q\in\mathbb{K}$ satisfying $|q|>1$. If $n<m$, we have*

$$(\log m-\log n)/n\log|q|\leq\rho(f)\leq\log(m+n-1)/\log|q|.$$

Theorem 4. *Let f be a non constant meromorphic solution of equation (2) and $q\in\mathbb{K}$ satisfying $|q|<1$. If $m\leq n$ then*

$$\rho(f)\leq\log(n/m)/-\log|q|.$$

1. Boutabaa A. Théorie de Nevanlinna p-adique (p-adic Nevanlinna theory). Manuscripta Math, 1990, 67, 3, 251–269.
2. Bouternikh S., Zerzaihi T. On some properties of ultrametric meromorphic solutions of Malmquist type. Russian Mathematics, 2022, 66, 8, 19–26.
3. Gundersen G.G., Heittokangas J., Laine I., Rieppo J., Yang D. Meromorphic solutions of generalized schröder equations. Aequationes Math, 2002, 63, 1, 110–135.
4. Heittokangas J., Laine I., Rieppo J., Yang D. Meromorphic solutions of some linear functional equations. Aequationes Math, 2000, 60, 1, 148–166.