Elliptic Problem With Nonlinear Singularity

Siham Boukarabila
University of Tlemcen, Tlemcen, Algeria

In this work we show the existence of solution for the following problem:

$$\begin{cases} -\Delta u &= \dfrac{|\nabla u|^q}{|u|^\alpha} + \lambda f \quad \text{in} \quad \Omega \\ u &= 0 \qquad\qquad\quad \text{on} \quad \partial\Omega \end{cases} \tag{1}$$

where $N > 2$, $\Omega \subset \mathbb{R}^N$ is a bounded regular domain, $q > 2$, $\alpha < \frac{q}{2}$, $\lambda > 0$ and f is a nonnegative function belonging to a suitable Lebesgue space.
Since $\alpha < \frac{q}{2}$, then problem (1) takes the form

$$\begin{cases} -\Delta u &= \frac{1}{(1-\beta)^q} |\nabla u^{1-\beta}|^q + \lambda f \quad \text{in} \quad \Omega \\ u &= 0 \qquad\qquad\qquad\qquad\quad \text{on} \quad \partial\Omega \end{cases} \tag{2}$$

where $\beta = \frac{\alpha}{q} < \frac{1}{2}$. Throughout this work, we will consider problem (2) as the main problem to be solved.

In ordre to solve problem (2) we established a Sobolev regularity for $v^{1-\beta}$ for all $\beta < \frac{1}{2}$, by analyzing v that is the unique solution for the Poisson problem below (3) and by using estimations on the Green's function.

$$\begin{cases} -\Delta v = f, & \text{in } \Omega, \\ v = 0, & \text{in } \partial\Omega, \end{cases} \tag{3}$$

Green's estimes: We gather, in the next lemma, some familiar estimates on the Green's function G and its gradient $\nabla_x G$.

Lemma 1. *Let $G(x,y)$ be the Green's Function associated to Ω , then there exist $\mathcal{C}_1 := \mathcal{C}_1(N,\Omega) > 0$ and $\mathcal{C}_2 := N$ such that*

$$G(x,y) \le \mathcal{C}_1 \min\left\{ \frac{1}{|x-y|^{N-2}}, \frac{\delta(x)}{|x-y|^{N-1}}, \frac{\delta(y)}{|x-y|^{N-1}} \right\}, \qquad \text{for a.e. } x, y \in \Omega,$$

and

$$|\nabla_x G(x,y)| \le \mathcal{C}_2\, G(x,y) \max\left\{ \frac{1}{|x-y|}, \frac{1}{\delta(x)} \right\}, \qquad \text{for a.e. } x, y \in \Omega.$$

Regularity result:

Theorem 1. *Suppose that $f \in L^m(\Omega)$, $m \ge 1$ is such that $f \gneq 0$ in Ω. Consider u to be the unique solution of the Poisson problem (3). Then for all $\beta \in (0, \frac{1}{2})$*

$$u^{1-\beta} \in W_0^{1,p}(\Omega), \qquad \forall\, p < \bar{p}(m)$$

where

$$\bar{p}(m) := \begin{cases} \dfrac{mN}{(N-m)(1-\beta) + Nm\beta}, & \text{if } m < N, \\ \dfrac{1}{\beta}, & \text{if } m \ge N. \end{cases}$$

Moreover, There exists a positive constant $C = C(N, m, p, \beta, \Omega)$ *such that*

$$\left\|u^{1-\beta}\right\|_{W_0^{1,p}(\Omega)} \leq C\|f\|_{L^m(\Omega)}^{1-\beta}$$

Existence result:

Theorem 2. *Assume that* $f \in L^m(\Omega), m \geq 1$ *is a nonnegative function. Then,*

1. *If* $\frac{N}{2} < m < N$ *and* $\beta < \frac{2m-N}{2(Nm-N+m)}$, *then for all* $2 < q < \frac{N}{(N-m)(1-\beta)+Nm\beta}$, *there exists* $\lambda^* = \lambda^*(N, q, m, f, \Omega) > 0$ *such that, for all* $0 < \lambda \leq \lambda^*$, *there exists a weak solution* u *to the problem (2) such that* $u^{1-\beta} \in W_0^{1,p}(\Omega)$ *for all* $1 \leq p < \frac{mN}{(N-m)(1-\beta)+Nm\beta}$.

2. *If* $m \geq N$ *and* $\beta < \frac{1}{2N}$, *then for all* $2 < q < \frac{1}{N\beta}$, *there exists* $\lambda^* = \lambda^*(N, q, m, f, \Omega) > 0$ *such that, for all* $0 < \lambda \leq \lambda^*$, *there exists a weak solution* u *to the problem (2) such that* $u^{1-\beta} \in W_0^{1,p}(\Omega)$ *for all* $1 \leq p < \frac{1}{\beta}$.

Uniqueness result:

Theorem 3. *Assume that* $f \in L^m(\Omega), m > N$ *is a nonnegative function such that* $f \gneq 0$. *Suppose that* $\beta < \frac{1}{2N}$ *and that* $2 < q < \frac{1}{N\beta}$. *Then problem (1) has at most one positive solution* u *such that* $u^{1-\beta} \in W_0^{1,p}(\Omega)$ *for all* $1 \leq p < \frac{1}{\beta}$.

1. S. Boukarabila, A. Primo, A. Younes. Singular elliptic problem with superquadratic growth in the gradient, Submitted.
2. B. Abdellaoui. Multiplicity Result for Quasilinear Elliptic Problems with General Growth in the Gradient. Journal: Adv. Nonlinear Stud. 8, no. 2, 2008, 289–301.
3. B. Abdellaoui, D. Giachetti, I. Peral and M. Walias. Non-linear elliptic problems with dependence on the gradient and singular on the boundary. Nonlinear Anal. 74, no. 4, 2011, 1355–1371.
4. B. Abdellaoui, D. Giachetti, I. Peral and M. Walias. Elliptic problems with nonlinear terms depending on the gradient and singular on the boundary: Interaction with Hardy-Leray potential. Discrete and Continue Dynamical System-A 34, no.5, 2014, 1747–1774.
5. N.E. Alaa, M. Pierre. Weak solutions of some quasilinear elliptic equations with data measures. SIAM J. Math. Anal. 24, 1993, 23–35.
6. H. Brezis, X. Cabré, Some simple nonlinear PDE's without solution. Boll. Unione. Mat. Ital. Sez. B, 8, 1998, 223–262.
7. Z.-Q. Chen and R. Song, Estimates on Green functions and Poisson kernels for symmetric stable processes. Math. Ann., 312(3), 1998, 465–501.