Thermodynamical Problem of High-Frequency Ultra Sound Waves Equation with Fractional Operators

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Nonlinear fractional partial differential equations (PDEs) have been used to model many phenomena in various fields such as mathematics, physics, and the evolution phenomena in different scientific areas. The property of the fractional derivative operators plays an especially crucial role in applied mathematics and physics, (Kilbas et al. 2006 [8], Diethelm 2010 [6]).

Exact solutions of fractional equations are used to mathematically formulate and, thus, aid in defining the solution of physical and other problems, including functions of several variables such as the propagation of heat or sound, etc. (see [1-5]).

Several mathematical models are used to describe nonlinear acoustics phenomena [7]. For example, In this work, we shall give a fractional model of nonlinear acoustics that is named the space-fractional Jordan-Moore-Gibson-Thompson (JMGT) equation. This equation results from modeling high-frequency ultra sound waves, and is written for $1 < \alpha \leq 2$ as follows

$$\begin{cases} \tau\psi_{ttt} + \mu\psi_{tt} - \kappa^2\partial_x^\alpha\psi - \delta\partial_x^\alpha\psi_t = F\left(x,t,\psi,\psi_x,\psi_{tt},\psi_{xx},(\psi_t)_{xx}\right), \\ \psi\left(\kappa t,t\right) = u_0\exp\left(-\frac{\kappa^2}{\delta}t\right), \ \psi_x\left(\kappa t,t\right) = \left(\psi_t\right)_x\left(\kappa t,t\right) = 0, \end{cases} \tag{1}$$

with

$$\partial_x^\alpha\psi = \begin{cases} \partial_x^2\psi, & \alpha = 2, \\ \mathcal{I}_{\kappa t}^{2-\alpha}\partial_x^2\psi = \frac{1}{\Gamma(2-\alpha)}\int_{\kappa t}^x (x-\tau)^{1-\alpha}\frac{\partial^2}{\partial\tau^2}\psi(\tau,t)\,d\tau, & 1<\alpha<2, \end{cases}$$

where the unknown scalar function $\psi = \psi(x,t)$ of a space and time variables $(x,t)\in\Omega$ with

$$\Omega = \left\{(x,t)\in\mathbb{R}\times[0,T];\ \kappa t\leq x\leq \ell\right\}, \text{ for } T>0 \text{ and } \ell\geq\kappa T,$$

denotes an acoustic velocity, where $\tau,\mu,\kappa,\delta\in\mathbb{R}_+^*$, $u_0\in\mathbb{C}$, also $F:\Omega\times\mathbb{C}\times\mathbb{C}\times\mathbb{C}\times\mathbb{C}\times\mathbb{C}\to\mathbb{C}$ is a nonlinear function.

The major goal of this work is to determine the existence and uniqueness for the fractional-order's problem of partial differential equation (1), under the traveling wave form

$$\psi(x,t) = \exp\left(-\frac{\kappa^2}{\delta}t\right)u(x-\kappa t), \text{ with } \kappa,\delta\in\mathbb{R}_+^*. \tag{2}$$

The basic profile u is not known in advance and is to be identified.

For the forthcoming analysis, we impose the following assumptions

$(A1)$ F is a continuous function that is invariant by the change of scale (2). It gives us

$$\begin{aligned} F\left(x,t,\psi,\psi_x,\psi_{tt},\psi_{xx},(\psi_t)_{xx}\right) &= \exp\left(-\frac{\kappa^2}{\delta}t\right)\times \\ &\quad \left(\delta\kappa f\left(\eta,u,u',u''\right) - \kappa^3\tau u'''\right), \end{aligned} \tag{3}$$

where $\eta = x-\kappa t$ and $f:[0,\ell]\times\mathbb{C}\times\mathbb{C}\times\mathbb{C}\to\mathbb{C}$ is a continuous function.

$(A2)$ There exist three positive constants $\beta,\gamma,\lambda>0$ so that the function f given by (3) satisfies

$$\left|f(\eta,u,v,w) - f(\eta,\bar{u},\bar{v},\bar{w})\right| \leq \beta\left|u-\bar{u}\right| + \gamma\left|v-\bar{v}\right| + \lambda\left|w-\bar{w}\right|, \ \forall\beta,\gamma,\lambda>0,$$

for each $\eta \in [0, \ell]$, and any $u, v, w, \bar{u}, \bar{v}, \bar{w} \in \mathbb{C}$.

$(A3)$ There exist four nonnegative functions $a, b, c, d \in C([0, \ell], \mathbb{R}_+)$, such that

$$|f(\eta, u, v, w)| \leq a(\eta) + b(\eta)|u| + c(\eta)|v| + d(\eta)|w|, \ \forall \eta \in [0, \ell],$$

for any $u, v, w \in \mathbb{C}$ and $\eta \in [0, \ell]$.

We denote by ϖ the positive constant defined by

$$\varpi = \max\left\{\frac{\ell(|q| + \gamma) + \alpha(|\theta| + \lambda)}{\ell^{1-\alpha}\Gamma(\alpha+1)}, \frac{\ell(|q| + c^*) + \alpha(|\theta| + d^*)}{\ell^{1-\alpha}\Gamma(\alpha+1)}\right\}.$$

Where $q = \frac{\kappa^2}{\delta^2}\left(\frac{3\tau\kappa^2}{\delta} - 2\mu\right)$, $\theta = \frac{\kappa}{\delta}\left(\frac{3\tau\kappa^2}{\delta} - \mu\right)$, and

$$a^* = \sup_{\eta\in[0,\ell]} a(\eta), \ b^* = \sup_{\eta\in[0,\ell]} b(\eta), \ c^* = \sup_{\eta\in[0,\ell]} c(\eta), \text{ and } d^* = \sup_{\eta\in[0,\ell]} d(\eta).$$

Throughout the rest of this paper, we put $p = \frac{\kappa^3}{\delta^3}\left(\frac{\tau\kappa^2}{\delta} - \mu\right)$.

Now, we give the principal theorems of this work.

Theorem 1. *Assume that the assumptions $(A1) - (A3)$ hold. If we put $\varpi \in (0,1)$ and*

$$\ell^{\alpha+1}\left(\frac{\kappa^3}{\delta^3}\left|\frac{\tau\kappa^2}{\delta} - \mu\right| + b^*\right) < \Gamma(\alpha+2)(1-\varpi),$$

then, there is at least one solution of the Cauchy problem (1) on Ω in the traveling wave form (2).

Theorem 2. *Assume that the assumptions $(A1)$, $(A2)$ hold. If we put $\varpi \in (0,1)$ and*

$$\frac{\ell^{\alpha+1}\left(\frac{\kappa^3}{\delta^3}\left|\frac{\tau\kappa^2}{\delta} - \mu\right| + \beta\right)}{\Gamma(\alpha+2)(1-\varpi)} < 1,$$

then the Cauchy problem (1) admits a unique solution in the traveling wave form (2) on Ω.

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