Asymptotic Representations of Rapidly Varying Solutions of Essentially Nonlinear Second Order Differential Equations with Regularly and Rapidly Varying Nonlinariaties

M. A. Belozerova[1], **O. O. Chepok**[2]

[1] Odesa I. I. Mechnykov National University, Odesa, Ukraine

[2]South Ukrainian National Pedagogical University named after K. D. Ushinsky, Odesa, Ukraine

marbel@ukr.net, olachepok@ukr.net

We consider the following differential equation

$$y'' = \alpha_0 p(t)\varphi_0(y')\varphi_1(y). \tag{1}$$

In this equation the constant α_0 is responsible for the sign of the equation, functions $p\colon [a,\omega[\to]0,+\infty[$, $(-\infty < a < \omega \le +\infty)$ and $\varphi_i : \Delta_{Y_i} \to]0,+\infty[$ $(i \in \{0,1\})$ are continuous, $Y_i \in \{0,\pm\infty\}$, Δ_{Y_i} is the some one-sided neighborhood of Y_i.

We also suppose that φ_1 is a regularly varying as $y \to Y_1$ function of index σ_1 (see, [6, p. 10]), function φ_0 is twice continuously differentiable on Δ_{Y_0} and satisfies the next conditions

$$\varphi_0'(y') \neq 0 \ \text{ as } \ y' \in \Delta_{Y_0}, \quad \lim_{\substack{y'\to Y_0\\ y'\in\Delta_{Y_0}}} \varphi_0(y') \in \{0,+\infty\}, \quad \lim_{\substack{y'\to Y_0\\ y'\in\Delta_{Y_0}}} \frac{\varphi_0(y')\varphi_0''(y')}{\left(\varphi_0'(y')\right)^2} = 1. \tag{2}$$

It follows from the above conditions (2) that the function φ_0 and its derivative of the first order are rapidly varying functions as the argument tends to Y_0 [1]. So we have the second order differential equation that contains the product of a regularly varying function of unknown function and a rapidly varying function of its first derivative in its right-hand side. In previous works [2] we obtained results for this kind of the equation containing a rapidly varying function of unknown function and a regularly varying function of its first derivative.

The solution y of the equation (1), that is defined on the interval $[t_0,\omega[\subset [a,\omega[$, is called $P_\omega(Y_0,Y_1,\lambda_0)$-solution $(-\infty \le \lambda_0 \le +\infty)$, if the following conditions take place

$$y^{(i)} : [t_0,\omega[\longrightarrow \Delta_{Y_i}, \quad \lim_{t\uparrow\omega} y^{(i)}(t) = Y_i \ \ (i=0,1), \quad \lim_{t\uparrow\omega} \frac{(y'(t))^2}{y''(t)y(t)} = \lambda_0.$$

In the work we establish the necessary and sufficient conditions for the existence of $P_\omega(Y_0,Y_1,\lambda_0)$–solutions of the equation (1) in case $\lambda_0 = 1$ and find asymptotic representations of such solutions and its first order derivatives as $t\uparrow\omega$. The results for $P_\omega(Y_0,Y_1,\lambda_0)$–solutions of the equation (1) in case $\lambda_0 \in R\setminus\{0;1\}$ were obtained in the work [3].

According to the properties of such solutions (see, for ex. [4]) we have that each such solution and its first-order derivative are rapidly varying functions as $t\uparrow\omega$ and

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)y'(t)}{y(t)} = 1, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)y''(t)}{y'(t)} = 1. \tag{3}$$

Note that the function $y(t(y'))$, where $t(y')$ is an inverse function to $y'(t)$, is a regularly varying function of the index 1 as $y' \to Y_1$ $(y' \in \Delta_{Y_1})$. It follows from (3):

$$\lim_{y'\to Y_1} \frac{y'(y(t(y')))'}{y(t(y'))} = \lim_{y'\to Y_1} \frac{(y'(t(y')))^2}{y(t(y'))y''(t(y'))} = 1.$$

During the proof of the results, equation (1) is reduced by a special transformation to the equivalent system of quasilinear differential equations. The limit matrix of coefficients of this system has real eigenvalues of different signs. We obtain that for this system of differential equations all the conditions of Theorem 2.2 in [5] take place. According to this theorem, the system has a one-parameter family of solutions $\{z_i\}_{i=1}^{2} : [x_1, +\infty[\longrightarrow R^2$ $(x_1 \geq x_0)$, that tends to zero as $x \to +\infty$.

Any solution of the family gives raise to such the $P_\omega(Y_0, Y_1, 1)$-solution y of the equation (1) that, together with its first derivative, admit the following asymptotic representations as $t \uparrow \omega$

$$y'(t) = \Phi_1^{-1}(I_1(t))[1 + o(1)], \quad y(t) = \Phi_1^{-1}(I_1(t))\pi_\omega(t)[1 + o(1)], \tag{4}$$

where

$$\pi_\omega(t) = \begin{cases} t, & \text{as } \omega = +\infty, \\ t - \omega, & \text{as } \omega < +\infty, \end{cases}$$

$$\Phi_0(z) = \int_{A_\omega}^{z} \frac{ds}{|s|^{\sigma_1}\varphi_0(s)}, \quad \Phi_1(z) = \int_{A_\omega}^{z} \Phi_0(s)ds, \quad Z_1 = \lim_{\substack{z \to Y_1 \\ z \in \Delta_{Y_1}}} \Phi_1(z),$$

and in case $y_0^0 \lim\limits_{t \uparrow \omega} |\pi_\omega(\tau)| = Y_0$,

$$I(t) = \alpha_0 y_0^0 \cdot \int_{B_\omega^0}^{t} |\pi_\omega(\tau)|^{\sigma_1} p(\tau)d\tau, ///I_1(t) = \int_{B_\omega^1}^{t} \frac{I(\tau)\Phi_0^{-1}(I(\tau))}{\pi_\omega(\tau)} d\tau.$$

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