The Birth of Limit Cycles in a Family of Planar Piecewise Differential Systems having two concentric circles as switching manifold

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The importance of studying piecewise linear differential systems has grown in recent years, due to their applications. Like we can see the appearance of this kind of system in modeling many natural phenomena, as in physics, biology, economics, etc. It is well known, that the limit cycles play a main role in the study of qualitative theory of piecewise differential systems.

In most of the published papers that studied the limit cycles of piecewise differential systems formed by linear systems consider only two pieces, some of which include in [1–4] and the references they contain. The simplest classes of discontinuous piecewise linear differential systems in the plane are those separated by one straight line, and it remains an open problem to know if three is the maximum number of limit cycles that this class of systems can have.

Regarding the separation curve, it's important to ask: If the separation curve is not a straight line, what happens to the number of limit cycles? How does the maximum number of limit cycles changes depending on how many zones the separation curve creates?

In this work we investigate the maximum number of limit cycles for a family of discontinuous piecewise linear differential systems separated by two concentric circles and formed by linear Hamiltonian differential systems having saddles, by the uses of the first integrales of these systems. Whithout loss of generality, we can assume after an affine change in $\mathbb{R}^2$ if necessary, that such first circle is the unit circle $\mathbb{S}_1 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\}$ and $\mathbb{S}_2 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2=k, k>1\}$. The two concentric circles $\mathbb{S}_1$ and $\mathbb{S}_2$ devided the plane onto three regions or zones named by

$$Z_1 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2<1\},$$
$$Z_2 = \{(x,y) \in \mathbb{R}^2 : 1<x^2+y^2<k\},$$
$$Z_3 = \{(x,y) \in \mathbb{R}^2 : k<x^2+y^2\}.$$

In the next lemma, we will introduce the normal form of the linear Hamiltonian differential having a saddle as the equilibrium point.

Lemma 1. *Any arbitrary linear differential Hamiltonian system in $\mathbb{R}^2$ having a saddle as the equilibrium point can be written as*

$$\dot{x} = -bx - \gamma y + d, \quad \dot{y} = \alpha x + by + c, \tag{1}$$

with $\alpha \in \{0,1\}$ and $b, \gamma, c, d \in \mathbb{R}$. Moreover, if $\alpha = 0$ then $c = 0$, and if $\alpha = 1$ then $\gamma = b^2 - \omega^2$ with $\omega \neq 0$. A first integral of this system (1) is

$$H(x,y) = -\frac{\alpha}{2}x^2 - bxy - \frac{\gamma}{2}y^2 - cx + dy.$$

In this work we consider the discontinuous piecewise linear differential systems separated by two concentric circles and formed by linear Hamiltonian differential systems having saddles of the form (1)

$$\dot{x} = \quad -b_i x - \gamma_i y + d_i, \quad \dot{y} = \alpha x + b_i y + c_i, \quad \text{in } Z_i, \tag{2}$$

where $\alpha_i \in \{0,1\}$ and $b_i, \gamma_i, c_i, d_i \in \mathbb{R}$ with $i = 1, 2, 3$, having y the first integrals

$$H_i(x,y) = \ -\frac{\alpha_i}{2}x^2 - b_i xy - \frac{\gamma_i}{2}y^2 - c_i x + d_i y.$$

with $i = 1, 2, 3$, given in Lemma 1.

In order to obtain a limit cycle that intersects the circle $\mathbb{S}_1$ in the two points $A_j = (a_{1j}, a_{2j})$ and $B_j = (b_{1j}, b_{2j})$, and interesects the circle $\mathbb{S}_2$ in the two points $C_j = (c_{1j}, c_{2j})$ and $D_j = (d_{1j}, d_{2j})$ where $A_j \neq B_j$, these two points must satisfy the following system

$$\begin{aligned}
e_1 = &\ H_1(a_{1j}, a_{2j}) - H_1(b_{1j}, b_{2j}) = 0,\\
e_2 = &\ H_2(a_{1j}, a_{2j}) - H_2(d_{1j}, d_{2j}) = 0,\\
e_3 = &\ H_2(b_{1j}, b_{2j}) - H_2(c_{1j}, c_{2j}) = 0,\\
e_4 = &\ H_2(c_{1j}, c_{2j}) - H_2(d_{1j}, d_{2j}) = 0,\\
e_5 = &\ a_{1j}^2 + a_{2j}^2 = 1, \quad e_6 = b_{1j}^2 + b_{2j}^2 = 1,\\
e_7 = &\ c_{1j}^2 + c_{2j}^2 = k, \quad e_8 = d_{1j}^2 + d_{2j}^2 = k,
\end{aligned} \tag{3}$$

Then our main result concerning the maximum number of limit cycles for discontinuous piecewise differential system separated by the two concentric circles $\mathbb{S}_1$ and $\mathbb{S}_2$ and formed by three linear Hamiltonian saddles are given in the following Theorem.

Theorem 1. *For the discontinuous piecewise differential systems separated by two concentric circles $\mathbb{S}_1$ and $\mathbb{S}_2$ and formed by linear Hamiltonian saddles, the maximum number of limit cycles is at most three limit cycles.*

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