Existence and asymptotic behavior of solutions for a nonlinear elliptic problem involving the fractional p-Laplacian

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We study the existence and the asymptotic behavior of sequence of positive solutions to the following nonlocal elliptic problem

$$\begin{cases} (-\Delta)_p^s u + u^{m-1} & = & \lambda u^{q-1} & \text{in } \Omega, \\ u & > & 0 & \text{in } \Omega, \\ u & = & 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1}$$

where $s \in (0,1)$, $\Omega \subset \mathbb{R}^N$,($N > ps$) is bounded domain, $q \in [p, p_s^*)$, the real parameter $\lambda > 0$ and m sufficiently large such that

$$p < q \leq p_s^* < m$$

The operator $(-\Delta)_p^s$ is the fractional p-Laplacian defined by

$$(-\Delta)_p^s u(x) := \text{ P.V. } \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}} \, dy, \qquad s \in (0,1), \quad p \geq 2$$

if $p = 2$ is the linear fractional Laplacian $(-\Delta)^s$.

Motivated by [1] and the study of nonlocal problems involving fractional p-Laplacian operator [2], [3], we prove existence results in the fractional Sobolev space defined by

$$X^s := \Big\{ u \in W_0^{s,p}(\Omega) : \int_\Omega |u|^m < \infty \Big\}.$$

Applying variational methods. Indeed, we assume a different hypotheses on the value of q to prove the existence results:

- Case 1: $p < q < m$
- Case 2: $p < q \leq p_s^* < m$

Definition 1. We say that $u \in X^s(\Omega)$ is a weak solution of equation (1) if u satisfies

$$\int\int_{D_\Omega} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+ps}} \, dx \, dy + \int_\Omega u^{m-1} v = \lambda \int_\Omega u^{q-1} v, \forall v \in X^s(\Omega). \tag{1}$$

Now we state our first existence result as the following theorem.

Theorem 1. *Let $s \in (0,1)$, $N > ps$ and $p < q \leq p_s^*$. Then, there exists $\underline{\lambda} > \lambda_* > 0$ such that for each $\lambda > \underline{\lambda}$ there is $m_0 > p_s^*$ such that for every $m \geq m_0$, equation (1) has at least two positive nontrivial solutions $z_m, u_m \in X_m^s(\Omega)$ and $z_m \not\equiv u_m$.*

Also, we provide a crucial regularity result about L^∞ boundedness of weak solution for equation (1) to be able to study the asymptotic behavior of the solutions found when $m \to +\infty$. This asymptotic behavior will be determined by an interesting limit problem, given in the following Theorem.

Theorem 2. *Assume that $\lambda > \lambda_*$ and let $\{u_m\}_m$ (resp. $\{z_m\}_m$)be a sequence of solutions. Then, there exists $u, z \in \mathcal{K} := \{u \in W_0^{s,p}(\Omega) : 0 \leq u \leq 1\}$ such that $u_m \rightharpoonup u$(resp. $z_m \rightharpoonup z$) strongly in $W_0^{s,p}(\Omega)$ and strongly in every Lebesgue space. Moreover, there exists $\mathcal{G}_u, \mathcal{G}_z \in L^\infty(\Omega)$ and u, z satisfy*

$$\begin{cases} (-\Delta)_p^s w + \mathcal{G}_w & = & \lambda w^{q-1} & \text{in } \Omega, \\ w & \geq & 0 & \text{in } \Omega, \\ w & = & 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{2}$$

where $0 < \mathcal{G}_w \leq \lambda$, $\mathcal{G}_w \not\equiv 0$, and $\mathcal{G}_w(1-w) = 0$, a.e Ω.

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