On solvability of generic boundary-value problems

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Let a finite interval $(a,b) \subset \mathbb{R}$ and parameters $\{m,\, n+1,\, r,\, l\} \subset \mathbb{N}$, $1 \le p \le \infty$, be given. We consider linear boundary-value problem

$$(Ly)(t) := y^{(r)}(t) + \sum_{j=1}^{r} A_{r-j}(t) y^{(r-j)}(t) = f(t), \quad t \in (a,b), \tag{1}$$

$$By = c. \tag{2}$$

Here, $A_{r-j}(\cdot) \in (W_p^n)^{m\times m}$, $f(\cdot) \in (W_p^n)^m$, $c \in \mathbb{C}^l$, and linear continuous operator

$$B\colon (W_p^{n+r})^m \to \mathbb{C}^l \tag{3}$$

are arbitrarily chosen; $y(\cdot) \in (W_p^{n+r})^m$ is unknown.

The solutions of equation (1) fill the space $(W_p^{n+r})^m$ if its right-hand side $f(\cdot)$ runs through the space $(W_p^n)^m$. Hence, the condition (2) with operator (3) is generic condition for this equation. It includes all known types of classical boundary conditions and numerous nonclassical conditions containing the derivatives (in general fractional) of an order greater than r. If $l < r$, then the boundary conditions are underdetermined. If $l > r$, then the boundary conditions are overdetermined.

With the problem (1), (2), we associate the linear operator

$$(L,B)\colon (W_p^{n+r})^m \to (W_p^n)^m \times \mathbb{C}^l. \tag{4}$$

Theorem 1. *The linear operator* (4) *is a bounded Fredholm operator with index* $mr - l$.

Consider family of matrix Cauchy problems with the initial conditions

$$Y_k^{(r)}(t) + \sum_{j=1}^{r} A_{r-j}(t) Y_k^{(r-j)}(t) = O_m, \quad t \in (a,b),$$
$$Y_k^{(j-1)}(a) = \delta_{k,j} I_m, \qquad j \in \{1,\dots,r\}.$$

By $[BY_k]$, we denote the numerical $(m \times l)$ – matrix, in which j-th column is result of the action of B on j-th column of $Y_k(\cdot)$.

Definition 1. A block numerical matrix

$$M(L,B) := ([BY_0], \dots, [BY_{r-1}]) \in \mathbb{C}^{rm\times l}$$

is characteristic matrix to problem (1), (2).

It consists of r rectangular block columns $[BY_k(\cdot)] \in \mathbb{C}^{m\times l}$.

If $B = 0$, then $M(L,B) = O_{rm\times l}$ for all L.

Theorem 2. *The dimensions of kernel and cokernel of the operator* (4) *are equal to the dimensions of kernel and cokernel of matrix $M(L,B)$, respectively:*

$$\dim\ker(L,B) = \dim\ker\big(M(L,B)\big),$$
$$\dim\operatorname{coker}(L,B) = \dim\operatorname{coker}\big(M(L,B)\big).$$

Corollary 1. *The operator* (4) *is invertible if and only if $l = mr$ and the square matrix $M(L,B)$ is nondegenerate.*

Example 1. Consider a linear one-point boundary-value problem

$$Ly(t) := y'(t) + Ay(t) = f(t), \quad t \in (a,b),$$

$$By = \sum_{k=0}^{n-1} \alpha_k y^{(k)}(a) = c.$$

Here, A is a constant $(m \times m)$ – matrix, $f(\cdot) \in (W_p^{n-1})^m$, $\alpha_k \in \mathbb{C}^{l\times m}$, $c \in \mathbb{C}^l$, $y(\cdot) \in (W_p^n)^m$,

$$B\colon (W_p^n)^m \to \mathbb{C}^l, \quad (L,B)\colon (W_p^n)^m \to (W_p^{n-1})^m \times \mathbb{C}^l.$$

By $Y(\cdot) \in (W_p^n)^{m\times m}$ we denote the unique solution of the Cauchy matrix problem

$$Y'(t) + AY(t) = O_m, \quad t \in (a,b), \quad Y(a) = I_m.$$

Then the matrix-valued function $Y(\cdot)$ and its k-th derivative will have the following form:

$$Y(t) = \exp\big(-A(t-a)\big), \quad Y(a) = I_m;$$

$$Y^{(k)}(t) = (-A)^k \exp\big(-A(t-a)\big), \quad Y^{(k)}(a) = (-A)^k, \quad k \in \mathbb{N}.$$

Substituting these values into the boundary condition, we have

$$M(L,B) = \sum_{k=0}^{n-1} \alpha_k (-A)^k.$$

It follows from Theorem 1 that $ind(L,B) = ind(M(L,B)) = m - l$.
Therefore, by Theorem 2, we obtain

$$\dim\ker(L,B) = \dim\ker\left(\sum_{k=0}^{n-1} \alpha_k(-A)^k\right) = m - \operatorname{rank}\left(\sum_{k=0}^{n-1} \alpha_k(-A)^k\right),$$

$$\dim\operatorname{coker}(L,B) = -m + l + \dim\ker\left(\sum_{k=0}^{n-1} \alpha_k(-A)^k\right) = l - \operatorname{rank}\left(\sum_{k=0}^{n-1} \alpha_k(-A)^k\right).$$

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