On elliptic equations in L_p-Sobolev spaces of generalized smoothness

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We discuss the solvability of elliptic equations and elliptic boundary problems on compact C^∞-manifolds in the Sobolev spaces $H_p^{s,\varphi}$ whose generalized smoothness is given by the number $s \in \mathbb{R}$ and function parameter $\varphi \in \Upsilon$, with the number exponent p satisfying $1 < p < \infty$. The set Υ consists of all positive functions $\varphi \in C^\infty([1,\infty))$ that satisfy the following two conditions: (i) φ varies slowly at ∞ in the sense of Karamata, i.e. $\varphi(\lambda t)/\varphi(t) \to 1$ as $t \to \infty$ whenever $\lambda > 0$; (ii) for each $m \in \mathbb{N}$ there is a number $c_m > 0$ that $t^m|\varphi^{(m)}(t)| \leq \varphi(t)$ whenever $t \geq 1$.

By definition, the complex linear space $H_p^{s,\varphi}(\mathbb{R}^n)$ consists of all tempered distributions w on $\mathbb{R}^n$ such that $w_{s,\varphi} := F^{-1}[\langle\xi\rangle^s\varphi(\langle\xi\rangle)(Fw)(\xi)]$ belongs to the Lebesgue space $L_p(\mathbb{R}^n)$. Here, as usual, F denotes the Fourier transform, and $\langle\xi\rangle := (1+|\xi|^2)^{1/2}$ whenever $\xi \in \mathbb{R}^n$. The Banach space $H_p^{s,\varphi}(\mathbb{R}^n)$ is endowed with the norm $\|w, H_p^{s,\varphi}(\mathbb{R}^n)\| := \|w_{s,\varphi}, L_p(\mathbb{R}^n)\|$. If Ω is an open nonempty subset of $\mathbb{R}^n$, then the Banach space $H_p^{s,\varphi}(\Omega)$ consists, by definition, of the restrictions of all distributions $w \in H_p^{s,\varphi}(\mathbb{R}^n)$ to Ω and is endowed with the norm $\|v, H_p^{s,\varphi}(\Omega)\| := \inf \|w, H_p^{s,\varphi}(\mathbb{R}^n)\|$ where the infimum is taken over all $w \in H_p^{s,\varphi}(\mathbb{R}^n)$ such that $w = v$ in Ω.

Let M be a compact oriented manifold of class C^∞ and dimension $n \geq 1$. As usual, ∂M is the boundary of M, and $M^\circ := M \setminus \partial M$, we admitting the $\partial M = \emptyset$ case. Choose a finite collection of local maps $\pi_j : \overline{\Pi}_j \leftrightarrow U_j$ on M, with $j = 1,\ldots,\varkappa$. Here, $\{U_j\}$ is a covering of M by open sets, and each Π_j denotes either $\mathbb{R}^n$ or $\mathbb{R}^n_+ :=: \{(x', x_n) : x' \in \mathbb{R}^{n-1}, x_n > 0\}$, with $\overline{\Pi}_j$ being the closure of Π_j. Choose functions $\chi_j \in C^\infty(M)$, with $j = 1,\ldots,\varkappa$, such that $\chi_1 + \cdots \chi_\varkappa = 1$ on M and that $\operatorname{supp}\chi_j \subset U_j$. By definition, the complex linear space $H_p^{s,\varphi}(M^\circ)$ consists of all extendable distributions u on M° such that $(\chi_j u)\circ\pi_j \in H_p^{s,\varphi}(\Pi_j)$ for every $j \in \{1,\ldots,\varkappa\}$, with $(\chi_j u)\circ\pi_j$ denoting the image of the distribution $\chi_j u$ in the local chart π_j. This space is endowed with the norm

$$\|u, H_p^{s,\varphi}(M^\circ)\| := \|(\chi_1 u)\circ\pi_1, H_p^{s,\varphi}(\Pi_1)\| + \cdots + \|(\chi_\varkappa u)\circ\pi_\varkappa, H_p^{s,\varphi}(\Pi_\varkappa)\|.$$

Theorem 1. *Every space $H_p^{s,\varphi}(M^\circ)$ is complete and does not depend up to equivalence of norms on our choice of $\{\pi_j\}$ and $\{\chi_j\}$. If M° is a Euclidean domain, then both definitions of $H_p^{s,\varphi}(M^\circ)$ are equivalent.*

Consider the case where $\partial M = \emptyset$, and suppose that A is an elliptic classical pseudodifferential operator on $M = M^\circ$ of order $r \in \mathbb{R}$ (specifically, A may be an elliptic linear partial differential operator with coefficients of class $C^\infty(M)$).

Theorem 2. *The operator A is bounded and Fredholm on the pair of spaces $H_p^{s,\varphi}(M)$ and $H_p^{s-r,\varphi}(M)$ whenever $s \in \mathbb{R}$, $\varphi \in \Upsilon$, and $1 < p < \infty$. Its kernel lies in $C^\infty(M)$ and together with the index does not depend on s, φ, and p (then the index is zero if $\dim M \geq 2$.)*

Suppose that $n \geq 2$ and $\partial M \neq \emptyset$. If $s > 1/p$, then the traces space for $H_p^{s,\varphi}(M^\circ)$ coincides with the Besov space $B_p^{s-1/p,\varphi}(\partial M)$ of generalized smoothness. The latter is built on the base of the relevant space over $\mathbb{R}^{n-1}$ by the localization of ∂M in the same manner as those for $H_p^{s,\varphi}(M^\circ)$.

Recall (see, e.g., [1, p. 55]) the definition of the space $B^{\sigma,\varphi}_p(\mathbb{R}^{n-1})$, with $\sigma \in \mathbb{R}$. Choose a function $\beta_0 \in C^\infty(\mathbb{R}^{n-1})$ such that $\beta_0(y) = 3/2$ if $|y| \leq 1$ and that $\beta_0(y) = 0$ if $|y| \geq 2$. Given $0 \leq k \in \mathbb{Z}$, put $\beta_k(y) := \beta_0(2^{-k}y) - \beta_0(2^{1-k}y)$ whenever $y \in \mathbb{R}^{n-1}$. The functions β_k form a diadic C^∞-resolution of unity on $\mathbb{R}^{n-1}$. The complex linear space $B^{\sigma,\varphi}_p(\mathbb{R}^{n-1})$ consists, by definition, of all tempered distributions w on $\mathbb{R}^{n-1}$ such that $w_k := F^{-1}[\beta_k F w]$ belongs to $L_p(\mathbb{R}^{n-1})$ whenever $0 \leq k \in \mathbb{Z}$ and that

$$\|w, B^{\sigma,\varphi}_p(\mathbb{R}^{n-1})\|^p := \sum_{k=0}^{\infty} \bigl(2^{\sigma k}\varphi(2^{\sigma k})\|w_k, L_p(\mathbb{R}^{n-1})\|\bigr)^p < \infty.$$

This space is complete with respect to the norm $\|w, B^{\sigma,\varphi}_p(\mathbb{R}^{n-1})\|$. The linear space $B^{\sigma,\varphi}_p(\partial M)$ is define to consist of all distributions v on ∂M such that $(\chi_j v) \circ \widetilde{\pi}_j \in B^{\sigma,\varphi}_p(\mathbb{R}^{n-1})$ for every $j \in \{1, \ldots, \lambda\}$. Here, we suppose that $\Pi_j = \mathbb{R}^n_+$ iff $1 \leq j \leq \lambda$, and we let $\widetilde{\pi}_j$ stand for the restriction of π_j to the boundary of $\mathbb{R}^n_+$. The space $B^{\sigma,\varphi}_p(\partial M)$ is endowed with the norm

$$\|v, B^{\sigma,\varphi}_p(\partial M)\| := \|(\chi_1 u) \circ \widetilde{\pi}_1, B^{\sigma,\varphi}_p(\mathbb{R}^{n-1})\| + \cdots + \|(\chi_\lambda u) \circ \widetilde{\pi}_\lambda, B^{\sigma,\varphi}_p(\mathbb{R}^{n-1})\|.$$

Theorem 3. *Every space $B^{\sigma,\varphi}_p(\partial M)$ is complete and does not depend up to equivalence of norms on our choice of $\{\pi_j\}$ and $\{\chi_j\}$. If $\sigma > 0$, then the mapping $u \mapsto u \restriction \partial M$, where $u \in C^\infty(M)$, extends uniquely (by continuity) to a bounded trace operator $R : H^{\sigma+1/p,\varphi}_p(M^\circ) \to B^{\sigma,\varphi}_p(\partial M)$. In this case, $B^{\sigma,\varphi}_p(\partial M)$ coincides with $R(H^{\sigma+1/p,\varphi}_p(M^\circ))$, and we have the equivalence of the norms*

$$\|v, B^{\sigma,\varphi}_p(\partial M)\| \asymp \inf\bigl\{\|u, H^{\sigma+1/p,\varphi}_p(M^\circ)\| : u \in H^{\sigma+1/p,\varphi}_p(M^\circ), v = Ru\bigr\}.$$

Consider an elliptic boundary problem on M of the form

$$Lu = f \;\text{ on }\; M^\circ, \quad B_j u = g_j \;\text{ on }\; \partial M, \;\; j = 1, ..., q.$$

Here, L is a linear differential operator on M of even order $2q \geq 2$, whereas each B_j is a linear boundary differential operator on ∂M of order $m_j \geq 0$. All coefficients of L and B_j are complex-valued C^∞-functions on M and ∂M, resp. Put $m := \max\{m_1, \ldots, m_q\}$, the $m \geq 2q$ case being admissible.

Theorem 4. *The mapping $u \mapsto (Lu, B_1 u, \ldots, B_q u)$, where $u \in C^\infty(M)$, extends uniquely (by continuity) to a Fredholm bounded operator on the pair of spaces*

$$H^{s,\varphi}_p(M^\circ) \;\text{ and }\; H^{s-2q,\varphi}_p(M^\circ) \times \prod_{j=1}^{q} B^{s-m_j-1/p,\varphi}_p(\partial M)$$

whenever $1 < p < \infty$, $s > m + 1/p$, and $\varphi \in \Upsilon$. The kernel of this operator lies in $C^\infty(M)$ and together with the index does not depend on s, φ, and p.

Similar results hold true for Besov and Triebel–Lizorkin spaces of generalized smoothness, which gives affirmative answers to some questions raised by H. Triebel in [1, Section 1.3.3]. These results are obtained together with A. Murach [2]. Their versions hold true for multi-order (i.e. Agmon–Douglis–Nirenberg) matrix elliptic operators and boundary problems.

1. Triebel H. Bases in Function Spaces, Sampling, Discrepancy, Numerical Integration, Zürich: European Mathematical Society, 2010.
2. Anop A., Murach A. Interpolation spaces of generalized smoothness and their applications to elliptic equations, arXiv:2003.05360, 32 pp. (Ukrainian).