Weak solutions for the $p_i(x)$-Laplacian equations With Variable Exponents and Degenerate Coercivity and L^m data

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In this paper we are concerned with existence and regularity of distributional solution for a nonlinear anisotropic elliptic equations with $p_i(x)$ growth conditions, degenerate coercivity and L^m data with m being small. This paper is contributes to the generalization of the research results [1] and [2].

Let us consider the following problem

$$\begin{cases} -\sum_{i=1}^{N} D_i\left(a_i(x,u)|D_iu|^{p_i(x)-2}D_iu\right) + |u|^{s(x)-1}u = f(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{1}$$

Here Ω is a bounded open subset of $\mathbb{R}^N (N \geq 2)$, $f, u\colon \Omega \to \mathbb{R}$ and $a_i\colon \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathédory functions satisfying the following condition, for almost $x \in \Omega$ and for all $s \in \mathbb{R}$

$$\frac{\alpha}{(1+|s|)^{\gamma(x)}} \leq a_i(x,s) \leq \beta, \quad \forall i = 1, ..., N. \tag{2}$$

Where α, β are positive constants. Moreover we consider the continuous functions $\gamma\colon \overline{\Omega} \to (0, +\infty)$, $s\colon \overline{\Omega} \to (0, +\infty)$ and $p_i\colon \overline{\Omega} \to (1, +\infty)$ such that

$$\frac{N\overline{p}(x) - m\overline{p}(x)(1+\gamma^+)}{Nm(\overline{p}(x) - 1 - \gamma^+)} < p_i(x) < \frac{N\overline{p}(x) - m\overline{p}(x)(1+\gamma^+)}{N\overline{p}(x) - m\overline{p}(x)(1+\gamma^+) - Nm(\overline{p}(x) - 1 - \gamma^+)} \tag{3}$$

and $\gamma^+ = \max_{x\in\Omega} \gamma(x)$, for each $i = 1, ..., N$

$$\overline{p}(x) = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i(x)}, \quad \text{with} \ \ \overline{p}(x) < N.$$

In this article, we shall be concerned with the existence of distributional solutions for a class of nonlinear anisotropic elliptic equation, The main difficulty of the problem is that even if the differential operator

$$u \longmapsto -\sum_{i=1}^{N} D_i\left(a_i(x,u)|D_iu|^{p_i(x)-2}D_iu\right),$$

is well defined between $W_0^{1,\overrightarrow{p}(.)}(\Omega)$ and its dual $W^{-1,\overrightarrow{p}'(.)}(\Omega)$, it is not coercive on $W_0^{1,\overrightarrow{p}(.)}(\Omega)$: degenerate coercivity means that when $|u|$ is too big, $\frac{1}{(1+|u|)^{\gamma(.)}}$ goes to zero. Due to the lack of coercivity, the standard method for variational inequalities involving pseudo-monotone operators can't be applied even if the data f is sufficiently regular. To overcome this problem, we will proceed by approximation by means of truncatures in $a_i(x,u)$ to get a coercive differential operator $a_i(x, T_n(u_n))$. Also, we prove some a priori estimate on the solutions of these problems. Moreover, we pass to the limit in the strong L^1 sense in the nonlinearity operator $a_i(x,u)|D_iu|^{p_i(x)-2}D_iu$, and finally conclude that the approximate solution u_ϵ converge to a solution of (1). We assume that the condition (2) holds, and $f \in L^m(\Omega)$ such that

$$\frac{N\overline{p}(x)}{\overline{p}(x)\left(1+\gamma^+\right)+2N\left(\overline{p}(x)-1-\gamma^+\right)} < m < \frac{N\overline{p}(x)}{N\overline{p}(x)-(1+\gamma^+)(N-\overline{p}(x))}. \tag{4}$$

Definition 1. A function u is a distribution solution of problem (1) if

$$u \in W_0^{1,1}(\Omega) \text{ and } |u|^{s(.)} \in L^1(\Omega),$$

and

$$\int_\Omega a_i(x,u)|D_iu|^{p_i(x)-2}D_iuD_i\varphi\,dx + \int_\Omega |u|^{s(x)-1}u\varphi\,dx = \int_\Omega f\varphi\,dx,$$

for all $\varphi \in C_0^\infty(\Omega)$.

Theorem 1. *Let $f \in L^m(\Omega)$, where m as in (4). Assuming $p_i(.), s(.)$ and $\gamma(.)$ are continuous functions such that for all $x \in \overline{\Omega}$*

$$s(x) \geq p_i(x), i = 1, ..., N,$$

and

$$0 < \gamma^+ < \overline{p}(x) - 1,$$

where $p_i(.)$ satisfying (3). Then the problem has a solution in the sense of distribution $u \in W_0^{1,\overrightarrow{q}(.)}(\Omega)$ where $q_i(.)$ are continuous functions on $\overline{\Omega}$ satisfying

$$1 \leq q_i(x) < \frac{Nmp_i(x)(\overline{p}(x)-1-\gamma^+)}{N\overline{p}(x)-m\overline{p}(x)(1+\gamma^+)}, \quad \forall x \in \overline{\Omega},\ \forall i = 1, ..., N.$$

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2. Ayadi H., Mokhtari F. Nonlinear anisotropic elliptic equations with variable exponents and degenerate coercivity. Elec. J. of Diff. Equa., 2018, No. 45, 1–23.