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THE BRACHISTOCHRONE PARAMETRIC CURVES THE SOLUTION IN THE NON-SINGULAR CASE $k > 0$

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The brachistochrone problem was formulated first by Johnn Bernoulli (1696,1697) in a chalenge to the mathematical community of that time and solved by himself serval months later (after that, solved also by his brother, Jacob Bernoulli, by Newton, Leibnitz, Euler and other great mathematicians), is considered to mark the beginning of calculs of variations. The brachistochrone is an optimal curve that allows for the quickest descent way of an object to slide frictionlessly by the influence of a uniform gravitational force, its curve has the shortest travel time compared to other types of curves, this problem was solved theoretically by St.Mirica in [3] using dynamic programming in [1,2] without the representation of curves, the curves can be used to calculate the distance, speed and travel time with and without the influence of the gravitational force for each curve. The aim of this work is to present comparative parametric curves of the Brachistochrone, based on the theoretical study in [3]. Dynamic programming formulation of the brachistochrone problem: given $k = \frac{(v_0)^2}{2g} \geq 0$, for any $y = (y_1, y_2) \in Y_0$. Find

$$\inf \; C(x(.)) = \int_0^{t_1} \frac{\|x'(t)\|}{\sqrt{x_2(t)+k}} dt, \;\; \|v\| = \sqrt{(v_1)^2 + (v_2)^2},$$

subject to:

$$\begin{array}{l} x'(t) \in F(x(t)) \text{ a.e } ([0,t_1]), \\ x(.) \in \Omega_1(y) = \{x(.) \in AC; x(0) = y, \; x(t_1) = 0\} \\ x(t) \in Y_0 \; \forall t \in [0,t_1), \, x(t_1) \in Y_1, \end{array}$$

where g the constant of the acceleration of gravity, v_0 the initial velocity, defined by the following data:

$$\begin{array}{l} Y_0 = (0,+\infty) \times (-k,+\infty), \, Y_1 = \{(0,0)\}, \, g(0,0) = 0, \\ F(x) = \mathbb{R}^2 \; \forall x \in Y = Y_0 \cup Y_1, \, g_0(x,v) = \frac{\|v\|}{\sqrt{x_2+k}}, \, \Omega_a = \Omega_1 = AC. \end{array}$$

Next, the pseudo-Hamiltonian is given in our case by: $\mathcal{H}(x,p,u,v) = \langle p, v\rangle + \frac{\|v\|}{\sqrt{x_2+k}}$, then, the Hamilonian and the corresponding marginl multifunction are given by the formulas:

$$H(x,p) = \begin{cases} 0 & if \; \|p\| \leq 0, \\ -\infty & if \; \|p\| > 0, \end{cases} , \; \widehat{F}(x,p) = \begin{cases} \{(0,0)\} & if \; \|p\| < \frac{1}{\sqrt{x_2+k}} = N(x_2), \\ \emptyset & if \; \|p\| > N(x_2), \\ \{\mu.p; \mu \leq 0\} & if \; \|p\| = N(x_2), \end{cases}$$

the domain Z is finitely-stratified by the stratifcation $S_H = \{S_0, S_1\}$ defined by:

$$S_0 = \{(x,p) \in Y_0 \times \mathbb{R}^2; \|p\| < N(x_2)\}, \quad S_1 = \{(x,p) \in Y_0 \times \mathbb{R}^2; \|p\| = N(x_2)\},$$

the set of terminal transversality points in our case is given as:

$$Z_1^* = \left\{ \left((0,0), (q_1, q_2) \right); \|q\|^2 = \frac{1}{k} \text{ if } k > 0 \right\},$$

the Hamiltonian system on the stratum S_1 (for wish $\|p\| = \frac{1}{\sqrt{x_2+k}} = N(x_2)$):

$$\begin{cases} x_1' = -2(x_2+k), & x_1(0) = 0, \\ x_2' = -2(x_2+k)\frac{p_2}{p_1}, & x_2(0) = 0, \\ p_1' = 0, & p_1(0) = q_1 = \frac{1}{\sqrt{k}}\cos\theta, \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}), \\ p_2' = \frac{1}{p_1}(p_1^2 + p_2^2), & p_2(0) = q_2 = \frac{1}{\sqrt{k}}\sin\theta, \end{cases}$$

the solution of system in the form of maximal flows is given by the formulas:

$$\begin{cases} X_1(t,\theta) = k.\tan\theta - \frac{k(2t+\sin(2t2t\theta))}{2\cos^2\theta}, \\ X_2(t,\theta) = -k + \frac{k(1+\cos(2t+2\theta))}{2\cos^2\theta}, \\ P_1(t,\theta) = \frac{1}{\sqrt{k}}\cos\theta, \\ P_2(t,\theta) = \frac{1}{\sqrt{k}}\cos\theta.\tan(t+\theta), \; \forall t \in I(\theta) = (-\theta - \frac{\pi}{2}, 0], \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}). \end{cases}$$

Our objective is to present the parametric curves of the Brachistochrone for the first time by the dynamic programming method then, we take the values of θ as a geometric sequence: $\theta = -\frac{\pi}{2} + 10^{-4} + n.\frac{1}{10}$, $\forall n \in [0, 32]$, with $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $t \in (-\frac{\pi}{2} - \theta, 0]$. Our results of the parametric graph representation results of the brachistochrone curve, show that: the travel time, value of the curve, and speed along the curve will be evaluated and compared for each curve, a very small modification of θ will have an influence on the time, because all curves that are not the same height do not have the same velocity. Value of the curvature can be used to analyze the speed along the curve. A small modification of the parameters will change the distance, travel time, value of the curve, and the speed along curve.

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