Existence of solution of Caputo Fabrizio fractional differential equations with not instantaneous impulses using DarbobЂ™s fixed point theorem

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Fractional calculus is a branch of classical mathematics, which deals with the generalization of operations of differentiation and integration to fractional order. Although fractional derivation theory is a subject almost as old as classical calculus as we know it today, its origins date back to the late 17th century, when Newton and Leibniz developed the foundations differential and integral calculus. In particular, Leibniz introduced the symbol $\frac{d^n f}{dt^n}$ when he announced in a letter to the Hospital , the Hospital replied "What does mean $\frac{d^n f}{dt^n}$ if $n = \frac{1}{2}$"? This letter from Hopital, written in 1695, is today accepted as the first incident of what we call fractional derivation. Differential equations with instantaneous impulses are frequently used to describe mathematical simulations of real-world phenomena that experience rapid changes of state [2,3].

In this study, we proved the existence of solutions of the the Cauchy problem of Caputo Fabrizio fractional differential equations with not instantaneous impulses [1]. Applying the Darbo's fixed point theorem along with the technique of Kuratowski measure of non compactness yields the existence result. The same initial value problem for linear implicit fractional differential equations with non instantaneous impulses under the Caputo Fabrizio fractional derivative was already studied with Monch's fixed point theorem and Kuratowski measure of noncompactness. Finally, we concluded by providing an example to illustrate the applicability of the found result.

We will study the existence of the following initial value problems (1)

$$\begin{cases} {}^{CF}D^{\alpha}_{s_i} u(t) = f(t, u(t)), t \in (s_i, t_{i+1}], i = 0, \ldots, m,\ 0 < \alpha < 1, \\ \\ u(t) = g_i(t, u(t_i^-)), t \in (t_i, s_i], i = 1, \ldots, m, \\ \\ u(0) = u_0 \in E, \end{cases} \tag{1}$$

where $I_i = (s_i, t_{i+1}]$, $J_i = (t_i, s_i]$, $f : I_i \times E \to E,\ g_i : J_i \times E \to E$, $i = 1 \cdots$, are given continuous functions, $I = [0, T]$, $T > 0$, $0 = s_0 < t_1 \le s_1 \le t_2 < \cdots < t_m \le s_m \le t_{m+1} = T$.

Theorem 1 (Darbo's fixed point Theorem)**.** *Let D be a non-empty, closed, bounded and convex subset of a Banach space E and let N be a continuous mapping of D into itself such that for any non-empty subset C of D,*

$$\mu(N(C)) \le k\mu(C)$$

Let us list some conditions on the functions involved in this IVP .

(H1) There exists a continuous function $G \in C(I_i, \mathbb{R}_+), i = 0 \cdots m$, such that

$$\|f(t, u)\| \le G(t)\|u\|,\ u \in E,\ t \in I_i \ \text{ with} G^* = \sup_{t \in \mathcal{I}} G(t).$$

(H2) There exists a continuous function $H_i \in C(J_i, \mathbb{R}_+), i = 1 \cdots m$, such that

$$\|g_k(t,u)\| \leq H_i(t),\ u \in E,\ t \in J_i \text{ with } H^* = \max_{i=0\ldots m}(\sup_{t\in J_i} H_i(t)).$$

(H3) For each bounded set $D \in E$ and for each $t \in I_i, i = 0 \cdots m$, we have

$$\mu(f(t,D)) \leq G(t)\mu(D),\ t \in I_i.$$

(H4) For each bounded set $D \in E$ and for each $t \in J_i, i = 1 \cdots m$, we have

$$\mu(g_i(t,D)) \leq H_i(t)\mu(D),\ t \in J_i.$$

Theorem 2. *Assume that assumptions $(H1) - (H4)$ hold. If*

$$k = \max\{H^*, (a_\alpha + Tb_\alpha)G^*\} < 1,$$

then the IVP (1) has at least one solution on I.

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