A minimization problem of an integral functional in Hilbert space

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We investigate here, a minimization problem of an integral functional, in a real separable Hilbert space, with a non-convex integrand in the control over the solutions to the control system described by first-order differential inclusions with mixed non-convex constraints. The new is that we involve a differential inclusion with maximal monotone operators. Along with the initial problem, we consider the relaxed problem with the convexicated control constraint and the convexicated integrand with respect to the control. Under suitable assumptions, we show that the relaxed problem has an optimal solution, and for any optimal solution, there exists a minimizing sequence of the initial problem converging to the optimal solution with respect to the trajectories and the functional. The papers [1], [3], [4], [5] are useful in our study.

Notations and definitions: Let H be a real separable Hilbert space. Let us give some definitions and properties of maximal monotone operators, see eg [2]. Define the domain, range and graph of a set-valued operator $A : D(A) \subset H \rightrightarrows H$ by

$$D(A) = \{x \in H : \ Ax \neq \emptyset\},$$
$$R(A) = \{y \in H : \ \exists\, x \in D(A), \ y \in Ax\} = \cup\{Ax : \ x \in D(A)\},$$
$$Gr(A) = \{(x, y) \in H \times H : \ x \in D(A), \ y \in Ax\}.$$

The operator $A : D(A) \subset H \rightrightarrows H$ is said to be monotone, if for $(x_i, y_i) \in Gr(A)$, $i = 1, 2$ one has $\langle y_1 - y_2, x_1 - x_2\rangle \geq 0$. It is maximal monotone, if its graph could not be contained strictly in the graph of any other monotone operator, in this case, for all $\lambda > 0$, $R(I_H + \lambda A) = H$, where I_H denotes the identity map of H.

If A is a maximal monotone operator, then, for every $x \in D(A)$, Ax is non-empty, closed and convex. Then, the projection of the origin onto Ax, denoted A^0x, exists and is unique.

Define for $\lambda > 0$, the resolvent of A by $J^A_\lambda = (I_H + \lambda A)^{-1}$ and the Yosida approximation of A by $A_\lambda = \frac{1}{\lambda}\left(I_H - J^A_\lambda\right)$. These operators are both single-valued and defined on the whole space H, and one has

$$J^A_\lambda x \in D(A) \ \text{ and } \ A_\lambda x \in A(J^A_\lambda x) \text{ for every } \ x \in H,$$

$$||A_\lambda x|| \leq ||A^0 x|| \ \text{ for every } \ x \in D(A).$$

Formulation of the problem: We study the relationship between the solutions of the problem

$$\inf \int_0^1 k(t, x(t), u(t))dt, \tag{1}$$

where $(x(\cdot), u(\cdot))$ is a solution of the following control system

$$(P)\begin{cases} -\dot{x}(t) \in A_t x(t) + B(t, x(t))u(t) & \text{a.e. } t \in I = [0, 1], \\ x(0) = x_0 \in D(A_0), \end{cases}$$

where $A_t : D(A_t) \subset H \rightrightarrows H$ is a maximal monotone operator, $B : I \times H \to \mathcal{L}(Y, H)$ where $\mathcal{L}(Y, H)$ denotes the space of continuous linear operators from Y to H, and the relaxed problem

$$\inf \int_0^1 k_U^{**}(t, x(t), u(t))dt, \tag{3}$$

where $k_U^{**}(t, x(t), u(t))$ is the bipolar of the function $u \mapsto k_U(t, x(t), u(t))$ which is defined as follows

$$k_U(t, x, u) = \begin{cases} k(t, x, u) & u \in U(t, x) \\ +\infty, & \text{otherwise,} \end{cases}$$

and $(x(\cdot), u(\cdot))$ is the solution of the control system (P).
Now, we address our main theorem as follows:

Theorem 1. *Under suitable assumptions, the relaxed problem (3) has an optimal solution $(x_*(\cdot), u_*(\cdot))$ and for any optimal solution there is a minimizing solution $(y_n(\cdot), v_n(\cdot))$ of problem (1), where $y_n(\cdot)$ converges to $x_*(\cdot)$.*

Moreover, there exists a subsequence $(y_{n_k}(\cdot), v_{n_k}(\cdot))$ of the minimizing sequence $(y_n(\cdot), v_n(\cdot))$, where $y_{n_k}(\cdot)$ converges to $x_*(\cdot)$.

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