Multi-valued differential equation with applications

A. Bouach[1], T. Haddad [2], L. Thibault[3]
[1]University of Mohammed Seddik Ben Yahia, Jijel, Algeria
[2]University of Mohammed Seddik Ben Yahia, Jijel, Algeria
[3]Montpellier University, Institute Alexander Grothendieck 34095, Montpellier, France
abderrahimbouach@gmail.com, haddadtr2000@yahoo.fr, lionel.thibault@umontpellier.fr

The sweeping processes was introduced and deeply studied by J. J. Moreau in the from of the discontinuous differential inclusion

$$(SP): \begin{cases} -\dot{x}(t) \in F_0(t, x(t)), \quad a.e.\ t \in [T_0, T], \\ F_0(t, x(t)) = N_{C(t)}(x(t)), \\ x(T_0) = x_0 \in C(T_0), \end{cases}$$

where $T_0, T \in \mathbb{R}$ with $0 \leq T_0 < T$ and $N_{C(t)}(\cdot)$ denotes here the normal cone of $C(t)$ in the sense of convex analysis. After, sweeping process involving integral perturbation, i.e.,

$$\begin{cases} -\dot{x}(t) \in F_{f_1, f_{0,2}}(t, x(t)), \text{ a.e. } t \in [T_0, T], \\ F_{f_1, f_{0,2}}(t, x(t)) = N_{C(t)}(x(t)) + f_1(t, x(t)) + \displaystyle\int_{T_0}^{t} f_{0,2}(s, x(s))ds, \\ x(T_0) = x_0 \in C(T_0), \end{cases}$$

was considered earlier by Brenier et al. (JMAA 2013), and recently by Colombo and Kozaily (JCA 2020), where the integral perturbation $f_{0,2}(\cdot, \cdot)$ depends on one time variable s.

In this talk, we analyze and discuss the well-posedness of a new differential variant

$$(P_{f_1, f_2}): \begin{cases} -\dot{x}(t) \in F_{f_1, f_2}(t, x(t)), \text{ a.e. } t \in [T_0, T], \\ F_{f_1, f_2}(t, x(t)) = N_{C(t)}(x(t)) + f_1(t, x(t)) + \displaystyle\int_{T_0}^{t} f_2(t, s, x(s))ds, \\ x(T_0) = x_0 \in C(T_0). \end{cases}$$

In this variant the normal cone of the moving set $C(t)$ is perturbed by a sum of a Carathéodory mapping and an integral forcing term. The integrand of the forcing term depends on two time-variables, that is, we study a general integro-differential sweeping process of Volterra type. By setting up an appropriate semi-discretization method combined with a new Gronwall-like inequality (differential inequality) [1, Lemma 3.3], we show that (P_{f_1, f_2}) has one and only one absolutely continuous solution. We also established the continuity of the solution with respect to the initial value. The results of this problem are applied to a frictionless mechanical problem.

We give our main results in the study of the integro-differential sweeping process (P_{f_1, f_2}).

Theorem 1. *[1, Theorem 4.2] Let H be a real Hilbert space. Then for any initial point $x_0 \in H$, with $x_0 \in C(T_0)$ there exists a unique absolutely continuous solution $x : [T_0, T] \longrightarrow H$ of the differential inclusion (P_{f_1, f_2}) under natural and mild assumptions.*

For the proof, we have construct a sequence of maps $x_n(\cdot) \in \mathcal{C}([T_0, T], H)$ which by Cauchy property, one can construct a limit mapping (when the length of subintervals tends to zero) which satisfies the desired integro-differential inclusion (P_{f_1, f_2}).

Now, we give the following stability result, if the initial data of the prolem x_0 change slightly, then the corrosponding solutions would not differ much.

Proposition 1. *[2, Proposition 3.1] For each $a \in C(T_0)$, denote by $x_a(\cdot)$ the unique solution of the integro-differential sweeping process*

$$\begin{cases} -\dot{x}(t) \in N_{C(t)}(x(t)) + f_1(t, x(t)) + \displaystyle\int_{T_0}^{t} f_2(t, s, x(s))\, ds \quad a.e \ \ in \ \ [T_0, T] \\ x(T_0) = a \in C(T_0). \end{cases}$$

Then, the map $\psi : a \longrightarrow x_a(\cdot)$ from $C(T_0)$ to the space $\mathcal{C}([T_0, T], H)$ endowed with the supremum norm (of uniform convergence) is Lipschitz on any bounded subset of $C(T_0)$.

We consider a quasistatic problem which models the contact between a deformable body and an obstacle, the so-called foundation. For our purpose of motivation, the main concern is to derive a formulation of the problem, expressed in terms of integro-differential sweeping process (P_{f_1,f_2}), and to prove its unique solvability under appropriate regularity hypotheses.

The formulation of the problem is as follows.

Problem 1. Find $u\colon \Omega \times [0, T] \to \mathbb{R}^d$ and $\sigma\colon \Omega \times [0, T] \to \mathbb{S}^d$ such that for a.e. $t \in [0, T]$

$$\sigma(t) = A\varepsilon(\dot{u}(t)) + B(t, \varepsilon(u(t))) + \int_0^t R(t-s)\varepsilon(u(s))\, ds \quad \text{in} \quad \Omega,$$

$$\operatorname{Div} \sigma(t) + f_0(t) = 0 \quad \text{in} \quad \Omega,$$

$$\sigma(t)\nu = f_N(t) \quad \text{on} \quad \Gamma_2,$$

$$u_\nu(t) \le 0, \ \ \sigma_\nu(t) \le 0, \ \ \sigma_\nu(t)u_\nu(t) = 0, \ \ \sigma_\tau(t) = 0 \quad \text{on} \quad \Gamma_3,$$

and

$$u(t) = 0 \quad \text{on} \quad \Gamma_1 \times [0, T],$$

$$u(0) = u_0 \quad \text{in} \quad \Omega.$$

Theorem 2. *Under an usual hypotheses of A, B, R and f_0, f_N, for each $u_0 \in U$, Problem 1 has a unique absolutely continuous solution $u(\cdot)$.*

The proof consists of two parts in which we rewrite Problem 1 in an equivalent form of integro-differential sweeping process (P_{f_1,f_2}) and apply the result of Theorem 1. To this end, we have used Green formula and Riesz representation theorem to obtain that any solution of Probem 1 is a solution of the integro-differential inclusion (P_{f_1,f_2}) where

$$H = \{v \in H^1(\Omega)^d : v = 0 \ \text{ on}\, \Gamma_1\}, \quad C(t) = \{v \in H : v_\nu \le 0 \quad \text{a.e. on} \ \ \Gamma_3\}.$$

$$f_1(t, v) = B(t, v) - f(t), \quad (B(t, v), w)_H = \int_\Omega B(t, \varepsilon(v))\varepsilon(w)\, dx, \ \text{ for all } v, w \in H, \ t \in [0, T].$$

$$f_2(t, s, v) = R(t-s)v, \quad (R(t)v, w)_H = \int_\Omega R(t)\varepsilon(v)\varepsilon(w), \ \text{ for all } v, w \in H, \ t \in [0, T].$$

1. Bouach A., Haddad T., Thibault L. Nonconvex Integro-Differential Sweeping Process with Applications. SIAM Journal on Control and Optimization, 2022, 60, No. 5, 2971–2995.
2. Bouach A., Haddad T., Thibault L. On the Discretization of Truncated Integro-Differential Sweeping Process and Optimal Control. J Optim Theory Appl, 2022, 193, 785–830.