On anisotropic degenerate parabolic problem with variable exponents

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This work is devoted to studying the existence and regularity of solutions for some nonlinear parabolic equations with principal part having degenerate coercivity:

$$\begin{gathered}
\partial_t u - \sum_{i=1}^{N}\left(D_i\left(\frac{|D_i u|^{p_i(x)-2}D_i u}{\Big(\ln(e+|u|)\Big)^{\sigma(x)}}\right)\right) + \sum_{i=1}^{N}|u|^{s_i(x)-1}u = f \quad \text{in } Q_T, \\
u = 0 \quad \text{on } \Sigma_T, \\
u(0,x) = u_0(x) \quad \text{in } \Omega.
\end{gathered} \tag{1}$$

where $Q_T \doteq (0,T)\times\Omega$, $\Sigma_T = (0,T)\times\partial\Omega$, Ω is a bounded open subset of $\mathbb{R}^N$, $(N \geq 2)$, $T > 0$ is a real number, $f \in L^{m(\cdot)}(Q_T)$, and $u_0 \in L^{(m(\cdot)-1)s_+(\cdot)+1}(\Omega)$, $s_+(\cdot) = \max_{1\leq i\leq N} s_i(\cdot)$. Here, we suppose that $Au := -\sum_{i=1}^{N}\left(D_i\left(\frac{|D_i u|^{p_i(x)-2}D_i u}{\left(\ln(e+|u|)\right)^{\sigma(x)}}\right)\right)$, where $m,\ \sigma \in C(\overline{\Omega})$, $m(\cdot) > 1$, $\sigma(\cdot) \geq 0$, and the variable exponents $p_i : \overline{\Omega} \longrightarrow (1,\infty)$, $s_i : \overline{\Omega} \longrightarrow (1,\infty)$ are continuous functions. The main difficulties in studying (1) are the fact that, the differential operator Au is not coercive if u is very large, and the problem (1) has a more complicated nonlinearity than the classical case $p_i(\cdot) = p_i$ since it is nonhomogeneous. This shows that the classical methods for the constant case [2] can't be applied here. In the classical case $\sigma = 0$ and $p_i(\cdot) = p_i$ the existence and regularity solution have been treated in [3]. It is worth pointing out that the problem (1) has been studied in [2] in the particular case $p_i(\cdot) = 2$, $i \in \{1,2,\cdots,N\}$, $m(\cdot) = m$, and $\sigma(\cdot) = \theta \in \left[0, 1+\frac{2}{N}\right)$ with $u_0 = 0$, where the authors have discussed the existence and regularity results based on (Lemma 2.2, [2]), but this technique does not work in the anisotropic case. Recently, in [4], the authors have studied the existence and regularity of weak solutions for the problem (1) in the whole $(0,T)\times\mathbb{R}^N$ with $p_i(\cdot)$ growth conditions and locally integrable data. In accordance with [1] our regularity results are new and have not been proven before neither in the isotropic nor in the anisotropic case.

Here we give an important definition which is essential to our study of the problem (1).

Definition 1. A function u is a weak solution of problem (1) if: $u \in L^1(0,T;W_0^{1,1}(\Omega)) \cap \Big(L^{s_+(\cdot)}(Q_T)\Big)$, and

$$\begin{aligned}
&-\int_0^T\int_\Omega u\partial_t\varphi\,dx\,dt + \sum_{i=1}^{N}\int_0^T\int_\Omega\left(\frac{|D_i u|^{p_i(x)-2}D_i u}{\Big(\ln(e+|u|)\Big)^{\sigma(x)}}\right)\cdot D_i\varphi\,dx\,dt \\
&+\sum_{i=1}^{N}\int_0^T\int_\Omega |u|^{s_i(x)-1}u\cdot\varphi\,dx\,dt = \int_0^T\int_\Omega \varphi(t,x)f\,dx\,dt + \int_\Omega \varphi(0,x)u_0(x)\,dx,
\end{aligned}$$

for all $\varphi \in C_c^1([0,T)\times\Omega)$, the C_c^1 functions with compact support.

Our main existence result for problem (1) is the following:

Theorem 1. *Let $m(\cdot) = m$, $\sigma(\cdot) = \sigma$, $f \in L^m(Q_T)$ with $m > 1$, such that*

$$\frac{(N+\sigma+2)\overline{p}(\cdot)}{(N+1)\overline{p}(\cdot) - 2N(1+\sigma) + (N+\sigma+2)\overline{p}(\cdot)} < m < \frac{(N+\sigma+2)\overline{p}(\cdot)}{(N+\sigma+2)\overline{p}(\cdot) - N(\sigma+1)},$$

$$0 \le \sigma < \min\left\{\overline{p}(\cdot) - 1 + \frac{\overline{p}(\cdot)}{N}, \overline{p}(\cdot) - 2 + \frac{m\overline{p}(\cdot)}{N}\right\},\ \overline{p}(\cdot) \ge 2.$$

Assume that $s_i(\cdot)$, $p_i(\cdot)$ are continuous functions such that for all $i = 1, \dots, N$

$$\frac{\overline{p}(\cdot)\Big(N+1-(1+\sigma)(m-1)\Big)}{m\Big((N+1)\overline{p}(\cdot) - N(1+\sigma)\Big)} < p_i(\cdot) < \frac{\overline{p}(\cdot)\Big(N+1-(1+\sigma)(m-1)\Big)}{mN(1+\sigma) - (m-1)(N+\sigma+2)\overline{p}(\cdot)},\ s_i(\cdot) \ge p_i(\cdot).$$

Under the above assumptions, the problem (1) *has at least one weak solution*

$$u \in \bigcap_{i=0}^{N} L^{q_i^-}\left(0,T; W_0^{1,q_i(\cdot)}(\Omega)\right),\ 1 \le q_i(\cdot) < \frac{mp_i(\cdot)}{\overline{p}(\cdot)}\left(\frac{(N+1)\overline{p}(\cdot) - N(1+\sigma)}{N+1-(1+\sigma)(m-1)}\right).$$

where $q_i(\cdot)$ are continuous functions on $\overline{\Omega}$ satisfying for all $i = 1, \dots, N$

The proof of Theorem needs several steps: First, we approximate the problem (1) with sequence of problems having smooth solutions $(u_n)_n$. Then, after deriving uniform estimates on u_n, we pass to the limit using a compactness results as in [1].

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