A new wide neighbourhood primal-dual algorithm for quadratic programming

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In this work we propose a new class of primal-dual path-following interior point algorithms for solving quadratic problems. At each iteration, the method would select a target on the central path with a large update from the current iterate, and then the Newton method is used to get the search directions, followed by adaptively choosing the step sizes.

We consider the following primal-dual pair of quadratic problems problems

$$(P) \quad \left\{ \min_x \left\{ \frac{1}{2} x^t Q x + c^t x, Ax = b, x \geq 0 \right\} \right.$$

$$(D) \quad \left\{ \max_{y,z} \left\{ b^t y - \frac{1}{2} x^t Q x, A^t y + z - Qx = c, x \geq 0, z \geq 0, y \in R^m \right\} \right.$$

where $Q \in \mathbb{R}^{n \times n}$ is a positive semidefinite symmetric matrix of order n, $b, y \in \mathbb{R}^m$, $x, z, c \in \mathbb{R}^n$ and $A \in \mathbb{R}^{m \times n}$ has full row rank $(rg(A) = m \leq n)$.

We note that the optimality conditions of the primal-dual pair (P) and (D) can be written in the following way:

$$\begin{cases} Ax = b, \\ A^t y + z - Qx = c, \\ xz = 0, \\ (x, z) \geq 0, \end{cases} \tag{1}$$

The basic idea of primal-dual IPMs is to replace the third equation in (1), the socalled complementary condition for (1), by the parameterized equation $xz = \mu e$, where e denotes the all-one vector and $\mu > 0$. This leads to the following system :

$$\begin{cases} Ax = b, \\ A^t y + z - Qx = c, \\ xz = \mu e, \\ (x, z) \geq 0, \end{cases} \tag{2}$$

Considering the approach introduced by Darvay [3], system (2) can be written in the following equivalent form:

$$\begin{cases} Ax = b, \\ A^t y + z - Qx = c, \\ \varphi(\frac{xz}{\tau\mu}) = \varphi(e), \\ (x, z) \geq 0, \end{cases} \tag{3}$$

where $\tau \in [0, 1]$ is the centering parameter and φ is a continuously differentiable function such that $\varphi(t) > 0$ for all $t > 0$.

Applying Newton's method to (3), for a given feasible solution (x, y, z) and considering $\varphi(t) = \sqrt{t}$ leads us to the following system for the search direction $(\Delta x, \Delta y, \Delta z)$:

$$\begin{cases} A\Delta x = 0 \\ A^T\Delta y + \Delta z - Q\Delta x = 0 \\ z\Delta x + x\Delta z = 2(\sqrt{\tau\mu xz} - xz) \end{cases} \tag{4}$$

The key ingredient of the method proposed by Ai and Zhang [1] is based on decomposing the Newton direction in positive and negative parts.

We consider the following two systems:

$$\begin{cases} A\Delta x_- = 0 \\ A^T\Delta y_- + \Delta z_- - Q\Delta x_- = 0 \\ z\Delta x_- + x\Delta z_- = 2(\sqrt{\tau\mu xz} - xz)^- \end{cases} \tag{5}$$

and

$$\begin{cases} A\Delta x_+ = 0 \\ A^T\Delta y_+ + \Delta z_+ - Q\Delta x_+ = 0 \\ z\Delta x_+ + x\Delta z_+ = 2(\sqrt{\tau\mu xz} - xz)^+ \end{cases} \tag{6}$$

Let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R} \times \mathbb{R}$; where $0 \le \alpha_1 \le 1$ and $0 \le \alpha_2 \le 1$, then

$$(x(\alpha), y(\alpha), z(\alpha)) = (x, y, z) + \alpha_1(\Delta x_-, \Delta y_-, \Delta z_-) + \alpha_2(\Delta x_+, \Delta y_+, \Delta z_+)$$

The best α is determined such that $(x(\alpha), y(\alpha), z(\alpha)) \in W(\beta, \tau)$ where

$$W(\beta, \tau) = \left\{(x, y, s) \in F^\circ / \left\|\left(\sqrt{\tau\mu e} - \sqrt{xz}\right)^+\right\| \le \sqrt{\beta\tau\mu}\right\}$$

With $\beta, \tau \in [0, 1]$.

1. Ai Wenbao, S.Zhang, An O($\sqrt{n}$L) iteration primal-dual path-following method, based on wide neighborhoods and large updates, for monotone LCP. SIAM Journal on Optimization, 16(2), 400–417, 2005.
2. J. Gondzio, Convergence analysis of an inexact feasible interior point method for convex quadratic programming. SIAM Journal on Optimization, Vol 23, No 3, 1510-1527, 2013.
3. Z. Darvay, P.R.Takács, Large-step interior-point algorithm for linear optimization based on a new wide neighborhood. Central European Journal of Operations Research, 26, 551–563, 2018.