Sampling numbers of smoothness classes via ℓ^1-minimization

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We study the recovery problem for functions $\Omega \to \mathbb{C}$ belonging to a known quasi-Banach smoothness space $\mathcal{F}$ given only m function values. The worst-case L^2-approximation error in this setup, i.e.,

$$\varrho_m(\mathcal{F})_{L^2} = \inf_{t_1,\dots,t_m \in \Omega} \inf_{R:\mathbb{C}^m \to L^2} \sup_{\substack{f \in \mathcal{F}: \\ \|f\|_{\mathcal{F}} \leq 1}} \|f - R(f(t_1), \dots, f(t_m))\|_{L^2},$$

is called the mth sampling number of $\mathcal{F}$. We derive new upper bounds for the sampling numbers through an explicit nonlinear recovery map R which is based on ℓ^1-minimization (basis pursuit denoising). In relevant cases such as mixed and isotropic weighted Wiener spaces or mixed-smoothness Sobolev spaces, sampling numbers in L^2 can be upper bounded by best n-term trigonometric widths in L^∞, i.e.,

$$\sigma_n(\mathcal{F}; \mathcal{B})_{L^\infty} = \sup_{\substack{f \in \mathcal{F}: \\ \|f\|_{\mathcal{F}} \leq 1}} \inf_{\substack{J \subseteq I, \#J \leq n, \\ (c_j)_{j \in J} \in \mathbb{C}^J}} \left\| f - \sum_{j \in J} c_j b_j \right\|_{L^\infty}$$

with $\mathcal{B} = (b_j)_{j \in I}$ being the Fourier basis.

With this method, a significant gain in the rate of convergence compared to recently developed linear recovery methods is achieved. In this deterministic worst-case setting we see an additional speed-up of $n^{-1/2}$ compared to linear methods in case of weighted Wiener spaces. For their quasi-Banach counterparts even arbitrary polynomial speed-up is possible. Surprisingly, our approach allows to recover mixed smoothness Sobolev functions from $S^r_p W$ on the d-torus with a logarithmically better rate of convergence than any linear method can achieve when $1 < p < 2$ and d is large.

This is joint work with Tino Ullrich (TU Chemnitz) and Felix Voigtlaender (KU Eichstätt–Ingolstadt).

1. Jahn T., Ullrich T., Voigtlaender F. Sampling numbers of smoothness classes via ℓ^1-minimization. arXiv preprint, 2022, arxiv:2212.00445.