A New error estimate of generalized Schwarz algorithm for a class of elliptiv variational inequalities

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Schwarz algorithms were proposed for proving the solvability of elliptic boundary value problems on domains which consist of two or more overlapping subdomains. The solution is approximated by an infinite sequence of functions which results from solving a sequence of elliptic boundary value problems in each of the subdomain. Pierre-Louis Lions was the first one to develop these algorithms, he introduced the parallel Schwarz algorithm for the propose of parallel computing see [3-5], the theory of Lions's algorithm is the genuine counterpart of the theory developed over the years for the Schwarz algorithm.

In this work, we mainly deal with the error analysis of generalized parallel Schwarz algorithm for the obstacle problem associated with the elliptic variational inequalitiy:

$$\begin{cases} \text{Find} \quad u \in K^g \quad \text{such that} \\ a(u, v-u) \geq (f, v-u), \forall v \in K^g. \end{cases} \tag{1}$$

Here, Ω is a bounded convex domain of $\mathbb{R}^2$ with suffciently smooth boundary $\partial\Omega$, K^g is the implicit convex set defined by

$$K^g = \{v \in H^1(\Omega) : v = g \text{ on } \partial\Omega,\ 0 \leq v \leq \Psi \text{ in } \Omega\}, \tag{2}$$

where $\Psi \in W^{2,\infty}(\Omega)$ is the obstacle and g is a regular function defined on $\partial\Omega$.
We also define the bilinear form, supposed continuous and strongly coercive

$$a(u, v) = \int_\Omega (\nabla u.\nabla v)dx, \tag{3}$$

and the linear form in $L^\infty(\Omega)$

$$(f, v) = \int_\Omega f(x).v(x)dx. \tag{4}$$

Let V_h be the finite element space consisting of continuous piecewise linear functions, r_h is the usual finite element restriction operator in Ω and π_h is an interpolation operator on $\partial\Omega$. We define the discrete counterpart of (1.1) by:

$$\begin{cases} \text{Find } u_h \in K_h^g \quad \text{such that} \\ a(u, v-u_h) \geq (f, v-u_h), \forall v \in K_h^g \end{cases} \tag{5}$$

where

$$K_h^g = \{\ v \in V_h : v = \pi_h g \text{ on } \partial\Omega,\ 0 \leq v \leq r_h\Psi \text{ in } \Omega\}, \tag{6}$$

The concept of a nonmatching finite elements grids used in this work consists of decomposing the whole domain Ω into m overlapping sub-domains and to discretize each subdomain by an independent finite element method, this kind of discretizations is very interesting since they can be applied to solve many practical problems which cannot be handled by global discretizations. This kind of a problem had also studied when the domain was split into two subdomains using the alternating Schwarz algorithm see [1, 2].

The principal result of this study is to prove the error estimate in L^∞-norm, making use of a geometrical convergence and the uniform convergence, this error contains an extra power in $|\log h|$ than expected where the constant c is independent of Schwarz iterate n.

Theorem 1. *Let $h = max(h_i, h_j)$, $i = \overline{1, m-1}$, $j = \overline{2, m}$ and $i < j$. Then, there exists a constant c independent of both h and n such that*

$$\|u_M - u_{Mh}^{n+1}\|_{L^\infty(\Omega_M)} \leq ch^2 |\log h|^3; \quad M = \overline{i, j} \tag{7}$$

So, we have established a convergence order of Schwarz algorithm for the obstacle problem when the domain Ω is split into m sub-domains. The approach developed in this work relies on the geometrical convergence and the error estimate between the continuous and discrete Schwarz iterates.

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