Uniform observability of wave equation by spectral collocation method

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Consider the wave equation on a square domain $\Omega = (-1,1)^d \subset \mathbb{R}^d$ $(d = 1,2)$ with a control f acting on one part of the boundary $\Gamma \subset \partial\Omega$ for some time $t \in (0,T)$:

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } Q = (0,T)\times\Omega \\ u = f & \text{on } (0,T)\times\Gamma \\ u = 0 & \text{on } (0,T)\times\partial\Omega\backslash\Gamma \\ u(0,x) = u^0(x),\ u_t(0,x) = u^1(x) & \text{in } \Omega. \end{cases} \tag{1}$$

Given any $f \in L^2((0,T)\times\Gamma)$ and some initial data $(u^0,u^1) \in L^2(\Omega)\times H^{-1}(\Omega)$, problem (1) has a unique solution $(u,u_t) \in C([0,T], L^2(\Omega)\times H^{-1}(\Omega))$.

It is also well-known that, if $T > T_0$ with T_0 sufficiently large and $\Gamma \subset \partial\Omega$ satisfies some geometric conditions (see [1], [2]), for any initial data $(u^0,u^1) \in L^2(\Omega)\times H^{-1}(\Omega)$ there exists a control $f \in L^2((0,T)\times\Gamma)$ such that the solution of (1) $u \in C([0,T]; H_0^2(\Omega)) \cap C^1([0,T]; L^2(\Omega)))$ can be driven to any final target. We assume without loss of generality that this final target is the equilibrium, i.e.

$$u(T,x) = u_t(T,x) = 0,\ x \in \Omega. \tag{2}$$

In this work, we focus on the numerical approximation of these boundary controls f. This problem has been extensively studied in the last decades with different numerical methods. In particular, it is well known that a discretization of system (1) with finite elements or finite differences schemes reduces the problem to a finite-dimensional control problem which is not uniformly controllable with respect to the discretization parameter. This means that the discrete control problem does not provide a bounded sequence of controls and therefore it is not possible to use this strategy to approximate the continuous control.

Here we propose a new numerical approach based on the spectral collocation method. For the background and details on this method as well as on other classes of spectral methods (Galerkin, collocation,...) we refer the reader to Canuto et al [3].

The novelty here is that we are able to prove the uniform observability inequality which is necessary to prove the uniform controllability, and therefore a convergent sequence of controls as $N \to \infty$, by adding an extra discrete boundary control that vanishes as $N \to \infty$. This provides an accurate approximation of the continuous control. The result relies on two key aspects: a uniform observability inequality for the associated discrete adjoint system and a detailed spectral analysis of the discrete low frequencies. The first property allows us to obtain the uniform boundedness of discrete controls while the second one is used to obtain the convergence of the discrete control to the continuous one.

The method we present here to obtain the uniform observability inequality is new and considers, instead of the discrete collocation system, the equivalent continuous error equation associated with the polynomial approximation (see [3]). This error equation is the same wave equation but with a non-homogeneous second-hand term, known as the error term. Therefore, the observability inequality can be derived using the same techniques as in the continuous model and we only have to estimate this extra error term. This is an important advantage of the method since it can be easily extended to more general equations (elasticity, fluid dynamics, etc.) and higher dimensions, as long as we consider rectangular domains.

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