Global existence and Blow up of solutions of Fractional evolution problem

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We consider the blow-up, and global existence of solutions to the following time–space fractional diffusion problem

$$ {}_0^C D_t^{\alpha} u(x,t) + (-\Delta)^{\beta/2} u(x,t) = {}_0J_t^{\gamma}\left(|u|^{p-1}u\right)(x,t), \quad x \in \mathbb{R}^N,\ t>0, \tag{1} $$

supplemented with the initial data

$$ u(x,0) = u_0(x) \in C_0(\mathbb{R}^N), \tag{2} $$

where $0<\alpha<1-\gamma$, $0<\gamma<1$, $0<\beta\le 2$, $p>1$, ${}_0J_t^{\gamma}$ denotes the left Riemann-Liouville fractional integral of order γ, ${}_0^C D_t^{\alpha}$ is the Caputo fractional derivative of order α and $(-\Delta)^{\beta/2}$ stands for the fractional Laplacian operator of order $\beta/2$. We show that if $p<1+\beta(\alpha+\gamma)/\alpha N$, then every nonnegative solution blows up in finite time, and if $p \ge 1+\beta(\alpha+\gamma)/\alpha N$ and $\|u_0\|_{L^{q_c}(\mathbb{R}^N)}$ is sufficiently small, where $q_c = \alpha N(p-1)/\beta(\alpha+\gamma)$, then the problem has global solutions. Finally, we give an upper bound estimate of the life span of blowing-up solutions

Using a fixed point argument, we prove the local existence to problem (1)–(2). First, we give the definition of the mild solution of (1)–(2).

Definition 1. Let $u_0 \in C_0(\mathbb{R}^N)$ and $T>0$. We say that $u \in C([0,T], C_0(\mathbb{R}^N))$ is a mild solution of (1)–(2), if u satisfies

$$ u(x,t) = P_{\alpha,\beta}(t)u_0(x) + \int_0^t (t-s)^{\alpha-1} S_{\alpha,\beta}(t-s)\, {}_0J_s^{\gamma}\left(|u|^{p-1}u\right)(x,s)\,\mathrm{d}s, \quad t\in[0,T]. \tag{3} $$

Theorem 1. *Given $u_0 \in C_0(\mathbb{R}^N)$, then there exists a maximal time $T_{\max} = T(u_0) > 0$ such that problem (1)–(2) has a unique mild solution u in $C([0,T_{\max}], C_0(\mathbb{R}^N))$. Furthermore, either $T_{\max}=+\infty$ or $T_{\max}<+\infty$ and $\|u\|_{L^\infty((0,t),C_0(\mathbb{R}^N))} \to +\infty$ as $t \to T_{\max}$. If, in addition, $u_0 \ge 0$, $u_0 \not\equiv 0$, then $u(t) \ge P_{\alpha,\beta}(t)u_0 > 0$ for $t \in (0,T_{\max})$. Moreover, if $u_0 \in L^r(\mathbb{R}^N)$ for some $r\in[1,\infty)$, then $u \in C([0,T_{\max}), L^r(\mathbb{R}^N))$.*

Then, based on the test function argument, we deal with the follwing blow up result of (1)–(2). We first give the following definition of weak solution.

Definition 2. Let $u_0 \in L^\infty_{loc}(\mathbb{R}^N)$, $0<\beta\le 2$, $0<\alpha<1-\gamma$, $0<\gamma<1$ and $T>0$. We say that $u \in L^p((0,T), L^\infty_{loc}(\mathbb{R}^N))$ is a weak solution of (1)–(2) if

$$ \begin{aligned} \int_0^T \int_{\mathbb{R}^N} {}_0J_t^{\gamma}\left(|u|^{p-1}u\right)\psi(x,t)\,\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{R}^N} u_0\, {}_t^C D_T^{\alpha}\psi(x,t)\,\mathrm{d}x\mathrm{d}t \\ = \int_0^T \int_{\mathbb{R}^N} u\,(-\Delta)^{\beta/2}\psi(x,t)\,\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{R}^N} u\, {}_t^C D_T^{\alpha}\psi(x,t)\,\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{4} $$

for every $\psi \in C^1([0,T], H^\beta(\mathbb{R}^N))$ with $\operatorname{supp}_x \psi \subset\subset \mathbb{R}^N$ and $\psi(.,T)=0$.

Moreover, we give the following lemma which proves that every mild solution of problem (1)–(2) is a weak solution.

Lemma 1. *Let $u_0 \in C_0(\mathbb{R}^N)$ and $u \in C([0,T], C_0(\mathbb{R}^N))$ be a mild solution of (1)–(2). Then u is also a weak solution of (1)–(2), for $0 < \beta \leq 2$, $0 < \alpha < 1 - \gamma$, $0 < \gamma < 1$ and all $T > 0$.*

Then, we have

Theorem 2. *Let $u_0 \in C_0(\mathbb{R}^N)$ and $u_0 \geq 0$, $u_0 \not\equiv 0$. If*

$$1 < p < 1 + \frac{\beta(\alpha + \gamma)}{\alpha N},$$

then every solution of problem (1)–(2) blows up in finite time.

Next, we show that the solutions of (1)–(2) exist globally.

Theorem 3. *Let $u_0 \in C_0(\mathbb{R}^N)$ and $u_0 \geq 0$, $u_0 \not\equiv 0$. If $p \geq 1 + \beta(\alpha+\gamma)/\alpha N$ and $\|u_0\|_{L^{q_c}(\mathbb{R}^N)}$ is sufficienlly small, where $q_c = \alpha N(p-1)/\beta(\alpha+\gamma)$, then solutions of (1)–(2) exist globally. Note that we can take $|u_0(x)| \leq C|x|^{-\beta(\alpha+\gamma)/(p-1)}$ instead of $u_0 \in L^{q_c}(\mathbb{R}^N)$.*

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