METRIC, FORWARD CONVEX-PRESERVING, AND CONTINUOUS MAPS BETWEEN GRAPHS

Y.-L. Dekhtiar, S. Kozerenko
National University of Kyiv-Mohyla Academy, Kyiv, Ukraine
yur-liubomysl.dekhtiar@ukma.edu.ua, sergiy.kozerenko@ukma.edu.ua

In our work we consider three types of maps between connected graphs: metric, forward convex-preserving (FCP), and continuous. In particular, we are interested in the relations between these classes of maps; that is, we aspire to find the conditions, when these classes coincide.

The *graph metric* $d_G : V(G) \times V(G) \to \mathbb{R}$ over a graph G is defined as the shortest distance between two vertices. A *metric interval* $[u, v]_G$ between two vertices u and v in a graph G is the set of all vertices w such, that $d_G(u, v) = d_G(u, w) + d_G(w, v)$. In other words, it is the union of all shortest paths between u and v.

Definition 1. A map $f : V(G) \to V(H)$ between two graphs G and H is called *metric* if for every pair of vertices $u, v \in V(G)$ the following inequality holds: $d_H(f(u), f(v)) \leq d_G(u, v)$.

As one can see from the definition, metric maps are "non-expanding" maps in the sense that the distances between the vertices cannot increase upon applying the map. One can notice that homomorphisms satisfy this condition. Hence, homomorphisms are a subclass of metric maps. The following proposition makes it more obvious.

Proposition 1. [1] *A map $f : V(G) \to V(H)$ is metric if and only if every edge $uv \in E(G)$ we have $d_H(f(u), f(v)) \leq 1$.*

One can also characterize metric maps in terms of their action on connected sets.

Proposition 2. [1] *A map between two graphs G and H is metric if and only if every connected subset of vertices of G is mapped into a connected set in H.*

For more results on metric maps between graphs see [2].

Definition 2. A map $f : V(G) \to V(H)$ is called *FCP* if the image $f(A)$ of every convex set $A \subseteq V(G)$ is convex in H.

Definition 3. A map $f : V(G) \to V(H)$ is called *continuous* if for all $u, v \in V(G)$ we have $[f(u), f(v)]_H \subseteq f([u, v]_G)$.

As mentioned earlier, we are interested in the relationship between these three types of maps. The following proposition shows that they have a lot in common.

Proposition 3. *Let f be a map between G and H. Then f – continuous $\implies$ f – FCP $\implies$ f – metric.*

Therefore, continuous maps are a subclass of FCPs, and the FCPs are in turn a subclass of metric maps. We investigated under which conditions it is possible to claim that the reverse holds. More specifically, we looked into the following problem: for which fixed graphs H are all metric maps $f : V(G) \to V(H)$ (for any G) continuous? For example, in [1] it was proved that this is the case for trees H. This inspired us to consider similar problems, which arise from changing classes of maps in the statement. We got the following results.

Theorem 1. *Let H be a graph. Then for all graphs G all metric maps $f : V(G) \to V(H)$ are continuous if and only if H is a block graph.*

In other words, if a map into a block graph is metric, than it must be continuous; and block graphs are the only ones to hold such property. It turns out that the dual question has a trivial answer.

Proposition 4. *Let G be a graph. Then for all graphs H all metric maps $f : V(G) \to V(H)$ are continuous if and only if G is a complete graph.*

And for which Gs (Hs) are all metric maps FCP? This question is answered in the subsequent results.

Theorem 2. *Let H be a graph. Then for all graphs G all metric maps $f : V(G) \to V(H)$ are FCP if and only if H is a block graph.*

Proposition 5. *Let G be a graph. Then for all graphs H all metric maps $f : V(G) \to V(H)$ are FCP if and only if G is a complete graph.*

It appears that substituting continuous maps with FCPs does not bring anything new. Yet replacing metric maps with FCPs does yield different results.

Theorem 3. *Let G be a graph. Then for all graphs H all FCP maps $f : V(G) \to V(H)$ are continuous if and only if G is an interval monotone graph, i.e. all G's metric intervals are convex.*

This theorem gives an alternative characterization of interval monotone graphs in terms of their behaviour with respect to FCPs and continuous maps.

It remains to answer the next question: for which graphs H all FCP maps $f : V(G) \to V(H)$ for arbitrary G are continuous? We believe that those are also block graphs, but we are yet to prove the hypothesis. One could make use of the forbidden graph characterization of block graphs, namely, that block graphs are diamond-free chordal graphs. We obtained the following partial result.

Proposition 6. *Let H be a chordal graph that has an induced diamond subgraph. Then there exists a graph G and a map $f : V(G) \to V(H)$ such that f is FCP, but not continuous.*

1. Kozerenko S. Linear and metric maps on trees via Markov graphs. Comment. Math. Univ. Carolin., 2018, 59, 173–187.
2. Zamfirescu T. Non-expanding mappings in graphs. Adv. Appl. Math. Sci., 2010, 6, Issue 1, 23–32.