Characteristic sets of a picture fuzzy set

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A direct generalization of Zadeh's fuzzy sets [5] and Atanassov's intuitionistic fuzzy sets [1] is the following notion of picture fuzzy sets.

Definition 1. [2] Let X be a non-empty set. A picture fuzzy set A on X is an object of the form $A = \{\langle x, \mu_A(x), \eta_A(x), \upsilon_A(x)\rangle \mid x \in X\}$, where $\mu_A(x) \in [0,1]$ is called "the degree of positive membership of x in A", $\eta_A(x) \in [0,1]$ is called "the degree of neutral membership of x in A" and $\upsilon_A(x) \in [0,1]$ is called "the degree of negative membership of x in A". μ_A, η_A and υ_A satisfy $\mu_A(x) + \eta_A(x) + \upsilon_A(x) \leq 1$, for any $x \in X$. The quantity $\pi_A(x) = 1 - (\mu_A(x) + \eta_A(x) + \upsilon_A(x))$ is called "the degree of refusal membership of x in A".

The set of all the picture fuzzy sets in the universe X will be denoted by $PFS(X)$.

Definition 2. [3,4] Consider the bounded lattice $(\mathbb{D}^*, \preceq, \curlywedge, \curlyvee, 0_{\mathbb{D}^*}, 1_{\mathbb{D}^*})$ where
$\mathbb{D}^* = \left\{x = (x_1, x_2, x_3) \in [0,1]^3, x_1 + x_2 + x_3 \leq 1\right\}$, the order relation $\preceq$ defined by $x \preceq y$ if and only if $(x_1 < y_1$ and $x_3 \geq y_3)$ or $(x_1 = y_1$ and $x_3 > y_3)$ or $(x_1 = y_1$, $x_3 = y_3$ and $x_2 \leq y_2)$, $1_{\mathbb{D}^*} = (1,0,0)$, $0_{\mathbb{D}^*} = (0,0,1)$, and for each $x, y \in \mathbb{D}^*$, $x \curlywedge y$ and $x \curlyvee y$ defined as follows:

$$x \curlywedge y = \begin{cases} x, & \text{if } x \preceq y, \\ y, & \text{if } y \preceq x, \\ (x_1 \wedge y_1, 1 - x_1 \wedge y_1 - x_3 \vee y_3, x_3 \vee y_3), & \text{otherwise.} \end{cases}$$

$$x \curlyvee y = \begin{cases} y, & \text{if } x \preceq y, \\ x, & \text{if } y \preceq x, \\ (x_1 \vee y_1, 0, x_3 \wedge y_3), & \text{otherwise.} \end{cases}$$

This set plays the role of a prototype of a picture fuzzy set, and the study of this set allows us to perform picture fuzzy sets operations using these of $\mathbb{D}^*$.

Definition 3. [2,4] Let $X \neq \emptyset$ and let $A, B \in PFS(X)$. Using the laws of $\mathbb{D}^*$, we define

- $A \subseteq B$ iff $(\mu_A(x) < \mu_B(x)$ and $\upsilon_A(x) \geq \upsilon_B(x))$ or $(\mu_A(x) = \mu_B(x)$ and $\upsilon_A(x) > \upsilon_B(x))$ or $(\mu_A(x) = \mu_B(x)$ and $\upsilon_A(x) = \upsilon_B(x)$ and $\eta_A(x) \leq \eta_B(x))$, for all $x \in X$.

- $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \eta_{A\cap B}(x), \upsilon_A(x) \vee \upsilon_B(x)\rangle \mid x \in X\}$,

  where $\eta_{A\cap B}(x) = \begin{cases} \eta_A(x) & \text{if } A \subseteq B, \\ \eta_B(x) & \text{if } B \subseteq A, \\ 1 - \mu_A(x) \wedge \mu_B(x) - \upsilon_A(x) \vee \upsilon_B(x), & \text{otherwise.} \end{cases}$

- $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \eta_{A\cup B}(x), \upsilon_A(x) \wedge \upsilon_B(x)\rangle \mid x \in X\}$,

  where $\eta_{A\cup B}(x) = \begin{cases} \eta_B(x) & \text{if } A \subseteq B, \\ \eta_A(x) & \text{if } B \subseteq A, \\ 0 & \text{otherwise.} \end{cases}$

- $\emptyset = \{\langle x, 0, 0, 1\rangle \mid x \in X\}$ and $X = \{\langle x, 1, 0, 0\rangle \mid x \in X\}$.

Among the most important notions in fuzzy set theory are the notions of support, kernel, cuts and fuzzy line of degree α of a fuzzy set, where $\alpha \in \mathbb{D}^*$. In the sequel, we generalize these notions to the notions of a picture fuzzy set with respect the order $\preceq$ in Definition 2.

In what follows we denote by $\mathbb{D}_0^* = \mathbb{D}^* - \{0_{\mathbb{D}^*}\}, \mathbb{D}_1^* = \mathbb{D}^* - \{1_{\mathbb{D}^*}\}$.

Definition 4. Let $X \neq \emptyset$ and let $A \in PFS(X)$. We define the following crisp subsets

1. The support of A: $S(A) = \{x \in X \mid \mu_A(x) > 0 \text{ or } (\mu_A(x) = 0 \text{ and } \upsilon_A(x) < 1)\}$.
2. The kernel of A: $Ker(A) = \{x \in X \mid \mu_A(x) = 1, \eta_A(x) = 0 \text{ and } \upsilon_A(x) = 0\}$.
3. The α-cut of A: for all $\alpha \in \mathbb{D}_0^*$, $A_\alpha = \{x \in X \mid (\mu_A(x) > \alpha_1 \text{ and } \upsilon_A(x) \leq \alpha_3) \text{ or } (\mu_A(x) = \alpha_1 \text{ and } \upsilon_A(x) < \alpha_3) \text{ or } (\mu_A(x) = \alpha_1,\ \upsilon_A(x) = \alpha_3 \text{ and } \eta_A(x) \geq \alpha_2)\}$.
4. The strong α-cut of A: for all $\alpha \in \mathbb{D}_1^*$, $A_\alpha^+ = \{x \in X \mid (\mu_A(x) > \alpha_1 \text{ and } \upsilon_A(x) \leq \alpha_3) \text{ or } (\mu_A(x) = \alpha_1 \text{ and } \upsilon_A(x) < \alpha_3) \text{ or } (\mu_A(x) = \alpha_1,\ \upsilon_A(x) = \alpha_3 \text{ and } \eta_A(x) > \alpha_2)\}$.
5. The picture fuzzy line of degree α of A: $L_\alpha(A) = \{x \in X \mid A(x) = \alpha\}$, for all $\alpha \in \mathbb{D}^*$.

Proposition 1. *Let $X \neq \emptyset$ and let $A, B \in PFS(X)$. Then for all $\alpha, \beta \in \mathbb{D}^*$,*

1. $A_\alpha^+ \subseteq A_\alpha, L_\alpha(A) \subseteq A_\alpha,\ L_{1_{\mathbb{D}^*}}(A) = Ker(A)$.
2. *For all $\alpha \in \mathbb{D}_0^*$, $A \subseteq B$ iff $A_\alpha \subseteq B_\alpha$, and for all $\alpha \in \mathbb{D}_1^*$, $A \subseteq B$ iff $A_\alpha^+ \subseteq B_\alpha^+$.*
3. *For all $\alpha, \beta \in \mathbb{D}_0^*$, $\alpha \preceq \beta$ implies $A_\alpha \supseteq A_\beta$, and for all $\alpha, \beta \in \mathbb{D}_1^*$, $\alpha \preceq \beta$ implies $A_\alpha^+ \supseteq A_\beta^+$.*
4. $(A \cap B)_\alpha = A_\alpha \cap B_\alpha$, $(A \cup B)_\alpha \supseteq A_\alpha \cup B_\alpha$.
5. *$A = B$ if and only if $L_\alpha(A) = L_\alpha(B)$.*
6. *If $\alpha \neq \beta$, then $L_\alpha(A) \cap L_\beta(A) = \emptyset$.*
7. $L_\alpha(A) \cap L_\alpha(B) \subseteq L_\alpha(A \cap B) \subseteq A_\alpha \cap B_\alpha$, $L_\alpha(A) \cup L_\beta(B) \subseteq L_{\alpha \curlyvee \beta}(A \cup B)$.

Remark 1. The converse of (6) holds if $L_\alpha(A) \neq \emptyset$ or $L_\beta(A) \neq \emptyset$.

This theorem permit to express any picture fuzzy subset of X in terms of its α-cuts, strong α-cut and picture fuzzy line of degree α.

Theorem 1. *Let $X \neq \emptyset$ and let $A \in PFS(X)$. Then, for all $x \in X, \alpha, \lambda \in \mathbb{D}^*$:*
$A(x) = \underset{\alpha \in \mathbb{D}^*}{\curlyvee} \alpha A_\alpha(x) = \underset{\alpha \in \mathbb{D}^*}{\curlyvee} \alpha A_\alpha^+(x) = \underset{\alpha \in \mathbb{D}^*}{\curlyvee} \alpha L_\alpha(A)(x)$, *and* $A_\alpha = \underset{\alpha \preceq \lambda}{\cup} A_\lambda$.

1. Atanassov K. T. Intuitionistic fuzzy sets. Fuzzy Sets and Systems, 1986, 20, 1, 87 − 96.
2. Cuong B. C., Kreinovich V. Picture Fuzzy Sets-a new concept for computational intelligence problems. 2013 third world congress on information and communication technologies (WICT 2013). IEEE, 2013.
3. Klement E. P., Mesiar R., Stupňanová A. Picture fuzzy sets and 3-fuzzy sets. 2018 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE), IEEE. 2018.
4. Saadi D., Amroune A. More on picture fuzzy sets and theire properties. accepted in TWMS Journal of Applied and Engineering Mathematics.
5. Zadeh L. A. Fuzzy Sets. Information and Control, 1965, 8, 338 − 353.