Lie-orthogonal operators on metric Lie algebras

D. R. Popovych[1]

[1]Institute of Mathematics of National Academy of Sciences of Ukraine, Kyiv, Ukraine

[1]Memorial University of Newfoundland, St John's, Canada

dmytro.popovych@imath.kiev.ua

The condition of Lie orthogonality of operators defined on Lie algebras [1,2] naturally arises in the course of the study of certain structures connected to Lie algebras such as Kähler manifolds and Clifford structures. Here we extend the results obtained in [3] for semisimple and reductive algebras to general metric algebras, see [4, Chapter 3] for a detailed presentation .

Definition 1. A linear operator J on a Lie algebra $\mathfrak{g}$ is called *Lie-orthogonal* if $[Jx, Jy] = [x, y]$ for any $x, y \in \mathfrak{g}$.

Lie-orthogonal operators can be classified up to the following equivalence [3]:

Definition 2. We call Lie-orthogonal operators J and $\tilde{J}$ on a Lie algebra *equivalent* if the image of their difference is contained in the center of this algebra.

Lemma 1. *Let $\mathfrak{g}$ be a finite-dimensional centerless Lie algebra that decomposes into the direct sum of its ideals $\mathfrak{i}_1, \dots \mathfrak{i}_k$, $\mathfrak{g} = \mathfrak{i}_1 \oplus \cdots \oplus \mathfrak{i}_k$. An operator J is Lie-orthogonal on $\mathfrak{g}$ if and only if it can be represented as $J = J_1 \oplus \cdots \oplus J_k$, where for any $i = 1, \dots, k$ the operator J_i is Lie-orthogonal on $\mathfrak{i}_i$. If the center of the algebra $\mathfrak{g}$ is nonzero, then the same representation holds up to the equivalence of Lie-orthogonal operators.*

Recall that a Lie algebra $\mathfrak{g}$ is called a *metric algebra* if there exists a nondegenerate symmetric invariant bilinear form $B(x, y)$. Classical examples of metric Lie algebras are

- semisimple algebras, for each of which the respective Killing form is, up to a nonzero multiplier, the only such form,
- abelian algebras, for which every symmetric bilinear form is trivially invariant,
- reductive algebras as direct sums of semisimple and abelian ones.

For other examples of metric Lie algebras, see Example 1.

Lemma 2. *Any Lie-orthogonal operator on any metric Lie algebra is equivalent to one which is annuled by the polynomial $x^2 - 1$.*

Theorem 1. *Let $\mathfrak{g}$ be a finite-dimensional metric Lie algebra, and J a Lie-orthogonal operator on $\mathfrak{g}$. Then there exists a decomposition of $\mathfrak{g}$ into a direct sum of two ideals $\mathfrak{i}_+$, $\mathfrak{i}_-$: $\mathfrak{g} = \mathfrak{i}_+ \oplus \mathfrak{i}_-$ such that the operator J is of the form*

$$J = \mathrm{id}_{\mathfrak{i}_+} \oplus (-\mathrm{id}_{\mathfrak{i}_-}) + J_{\mathfrak{z}}, \tag{1}$$

where $J_{\mathfrak{z}}$ is an operator on $\mathfrak{g}$ with image in the center $\mathfrak{z}$.

Corollary 1. *Lie-orthogonal operators on any simple Lie algebra are exhausted by the trivial operators $\mathrm{id}_{\mathfrak{g}}$ ma $-\mathrm{id}_{\mathfrak{g}}$.*

Corollary 2. *Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra, and $\mathfrak{g} = \mathfrak{i}_1 \oplus \cdots \oplus \mathfrak{i}_p$ be its decomposition into simple components. Then any Lie-orthogonal operator J on $\mathfrak{g}$ is of the form $J = J_1 \oplus \cdots \oplus J_p$, where for every $s = 1, \dots, p$, either $J_s = \mathrm{id}_{\mathfrak{i}_s}$ or $J_s = -\mathrm{id}_{\mathfrak{i}_s}$.*

Combining Lemma 1 with Corollary 2 gives a complete description of Lie-orthogonal operators on reductive Lie algebras.

Corollary 3. *Let $\mathfrak{g}$ be a finite-dimensional reductive Lie algebra, and $\mathfrak{g}=\mathfrak{z}\oplus\mathfrak{i}_1\oplus\cdots\oplus\mathfrak{i}_p$ be its decomposition into center and simple components. An operator J on $\mathfrak{g}$ is Lie-orthogonal if and only if it has the form $J=J_0\oplus J_1\oplus\cdots\oplus J_p+J_{\mathfrak{z}}$, where for every $s=1,\dots,p$ either $J_s=\mathrm{id}_{\mathfrak{i}_s}$ or $J_s=-\mathrm{id}_{\mathfrak{i}_s}$, J_0 is a zero operator on $\mathfrak{z}$, $J_{\mathfrak{z}}$ is any operator on $\mathfrak{g}$ with image in the center $\mathfrak{z}$.*

Example 1. The four-dimensional non-abelian metric real Lie algebras are the two reductive algebras $\mathrm{A}_1\oplus\mathrm{sl}(2,\mathbb{R})$ and $\mathrm{A}_1\oplus\mathrm{so}(3)$ and two indecomposable solvable algebras, the oscillator algebra $\mathrm{A}^0_{4.9}$ and the diamond algebra $\mathrm{A}^{-1}_{4.8}$; see their commutation relations, e.g., in [4] or [5]. Theorem 1 implies that the set of Lie-orthogonal operators on each of these algebras is

$$\left\{\begin{pmatrix}a^1_1 & a^1_2 & a^1_3 & a^1_4\\ 0 & \varepsilon & 0 & 0\\ 0 & 0 & \varepsilon & 0\\ 0 & 0 & 0 & \varepsilon\end{pmatrix}\ \middle|\ \varepsilon=\pm1,\ a^1_1,a^1_2,a^1_3,a^1_4\in\mathbb{R}\right\}.$$

Remark 1. Metric Lie algebras do not exhaust all Lie algebras Lie-orthogonal operators on which necessarily admit the decomposition (1). Consider the nonmetric Lie algebra $\mathfrak{g}=\mathrm{sl}(2,\mathbb{C})\in_{\rho_1}2\mathfrak{g}_1$, where ρ_1 is the standard irreducible representation of $\mathrm{sl}(2,\mathbb{C})$ in the two-dimensional vector space. Up to the skew-symmetry of the Lie bracket, the nonzero commutation relations of this algebra in the canonical basis are exhausted by

$$[e_1,e_2]=2e_2,\quad [e_1,e_3]=-2e_3,\quad [e_2,e_3]=e_1,$$
$$[e_1,e_4]=e_4,\quad [e_2,e_5]=e_4,\quad [e_3,e_4]=e_5,\quad [e_1,e_5]=-e_5,$$

i.e., e_1, e_2 and e_3 constitute a basis of a Levi factor of $\mathfrak{g}$, which is isomorphic to $\mathrm{sl}(2,\mathbb{C})$, and (e_4,e_5) is a basis of the abelian radical $\mathfrak{r}$ of $\mathfrak{g}$. It is proved in [4, Proposition 3.53] that Lie-orthogonal operators on $\mathfrak{g}$ are exhausted by the trivial operators $\mathrm{id}_{\mathfrak{g}}$ та $-\mathrm{id}_{\mathfrak{g}}$.

Proposition 1. *Let J be a Lie-orthogonal operator on a finite-dimensional Lie algebra $\mathfrak{g}$ with the eigenvalue $\lambda=1$ and $\mathfrak{i}_1$ be the ideal corresponding to this eigenvalue. Then the restriction J_1 of the operator J on the ideal $\mathfrak{i}_1$ can be represented, up to equivalence of Lie-orthogonal operators on $\mathfrak{i}_1$, in the form $J_1=\mathrm{id}_{\mathfrak{i}_1}+N$, where N is a nilpotent operator on $\mathfrak{i}_1$ and the image of N is contained in the radical $\mathfrak{r}_1$ of the ideal $\mathfrak{i}_1$.*

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