On deformational properties of Morse functions on non-oriented surfaces

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Let M be a smooth compact surface and $D(M,Y)$ be the group of diffeomorphisms of M fixed on a closed subset $Y \subset M$. Consider the natural action of $D(M,Y)$ on the space of smooth functions $C^{\infty}(M,\mathbb{R})$ such that the result of the action of the diffeomorphism h on the function f is the composition $f \circ h$. Denote by

$$\mathcal{O}(f,Y) = \{f \circ h \mid h \in \mathcal{D}(M,Y)\}$$

the *orbit* of f under this action.

Definition 1. Let $\mathcal{P}$ be a minimal class of groups satisfying the following conditions:

1) $1 \in \mathcal{P}$;
2) if $A, B \in \mathcal{P}$, then $A \times B \in \mathcal{P}$;
3) if $A \in \mathcal{P}$ and $n \geq 1$, then $A \wr_n \mathbb{Z} \in \mathcal{P}$.

Definition 2. Let G, H be groups, $m \in \mathbb{Z}$ and $\gamma\colon H \to H$ be automorphism of order 2. Define $(G,H) \wr_{\gamma,m} \mathbb{Z}$ as a cartesian product $G^{2m} \times H^m \times \mathbb{Z}$ with the following operation:

$$\begin{aligned}
&(b_0,\dots,b_{2m-1},c_0,\dots,c_{m-1},k)(d_0,\dots,d_{2m-1},e_0,\dots,e_{m-1},l) = \\
&\quad = \big(b_0 d_k, b_1 d_{k+1}, \cdots, b_{2m-1} d_{k+2m-1}, \\
&\qquad c_0 e_k, c_1 e_{k+1}, \cdots, c_{m-1-k} e_{m-1}, c_{m-k}\gamma(e_0), \cdots, c_{m-1}\gamma(e_{k-1}),\ k+l\big),
\end{aligned}$$

where indices are taken modulo $2m$.

Theorem 1. *Let M be a Möbius band and let f be a Morse function. Then*

$$\pi_1\mathcal{O}(f,\partial M) \cong \pi_1\mathcal{O}(f|_{Y_0},\partial Y_0) \times$$
$$\times \Big(\prod_{i=1}^{d} \pi_1\mathcal{O}(f|_{D_i},\partial D_i), \prod_{j=1}^{e} \pi_1\mathcal{O}(f|_{E_j},\partial E_j)\Big) \wr_{\gamma,m} \mathbb{Z},$$

where Y_0, D_i, E_j are defined in [1] *subsurfaces of M being a cylinder and disks correspondingly.*

It was shown in [2] that if M has negative Euler characterictic, then fundamental groups of orbits of Morse functions are direct products of such groups for functions only on cylinders, disks and Möbius bands. Moreover, if M is either a 2-disk or a cylinder, then $\pi_1\mathcal{O}(f,\partial M) \in \mathcal{P}$. Together with this result Theorem 1 allows to compute fundamental groups of orbits of Morse functions on all non-oriented surfaces except Klein bottle and projective plane.

1. Kuznietsova I. V., Maksymenko S. I. Homotopy properties of smooth functions on the Möbius band. Proceedings of the International Geometry Center, 2019, vol. 12, no. 3, pp. 1-29.
2. Maksymenko S. I. Deformations of functions on surfaces by isotopic to the identity diffeomorphisms. Topology and its Applications, 2020, vol. 282.