Fractal dimension of the set of subsums for multigeometric series

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Let $M \subseteq N$. Then number

$$x = x\,(M) = \sum_{n \in M \subseteq N} u_n = \sum_{n=1}^{\infty} \varepsilon_n u_n, \quad \varepsilon_n = \begin{cases} 1 & \text{for} \;\; n \in M, \\ 0 & \text{for} \;\; n \notin M, \end{cases} \tag{1}$$

is called the subsum of the series $\sum u_n$. By $E(u_n)$ we denote the set of all subsums (1) for this series. This mathematical object is also defined by some scientists as achievement set of the sequence (u_n).

It is well known from [**?**] and [4] that the set of subsums for convergent positive series is one of the following three types: a finite union of closed intervals, homeomorphic to the Cantor set or M-Cantorval. However, the necessary and sufficient condition that the set of subsums is a Cantorval or homeomorphic to the Cantor set is still unknown. Despite essential progress for some series, the problem is quite difficult in general case. In this context scientists focus on series such that their terms are elements of some sequence (see [5] and [6]) with some condition of homogeneity (depend on finite numbers of parameters and defined by a formula for general term or recurrence relation).

Topological and metric properties of the set of subsums are investigating actively now for the series

$$\sum_{n=1}^{\infty} a_n = k_1 + k_2 + \dots + k_m + k_1 q + \dots + k_m q + \dots + k_1 q^n + \dots + k_m q^n + \dots, \tag{2}$$

where $k_1, k_2, \dots k_m$ are fixed positive integers, $q \in (0,1)$. Terms of this series are an elements of multigeometric sequence. In the papers [1] and [2] some conditions for $E(a_n)$ to be a Cantorval or a Cantor-type set were found. As a result of these articles we have the following theorem.

Theorem 1. *If series* (2) *satisfies the condition*

$$q < \frac{1}{cardA}, \tag{3}$$

where (cardA – cardinality of the set A)

$$A = \left\{ \sum_{i=1}^{m} \varepsilon_i k_i : (\varepsilon_i)_{i=1}^{m} \in \{0,1\}^m \right\},$$

then $E(a_n)$ is a Cantor-type set with zero Lebesgue measure.

However, there are no any results on calculation of Hausdorff-Besicovitch dimension of it.

We recall the definition and some basic properties of the fractal dimension. Let E be a bounded set in a metric space (X, ρ). We define $d(E)$ as a diameter for the set E

$$d(E) = \sup\{\rho(x,y) : x, y \in E\}.$$

Let ε be a fixed positive number. Finite or countable family of sets $\{E_j\}$ is called ε-covering of the set E if it satisfies the conditions

$$E \subset \bigcup E_j, \text{ where } d(E_j) \le \varepsilon, E_j \in X, \forall j \in N.$$

If α is some fixed positive numbers then $\alpha - \varepsilon$ measure of Hausdorff of a bounded set E we defined as follows

$$H_\varepsilon^\alpha(E) \equiv \inf_{d(E_j)\le\varepsilon} \{d(E_j)^\alpha\},$$

where we take infimum by all possible finite or countable ε-covering $\{E_j\}$ of the set E $(E_j \in X)$.

α - Hausdorff measure ($H^\alpha - measure$) of a bounded set E is defined as the value of function

$$H^\alpha(E) \equiv \lim_{\varepsilon\to 0} H_\varepsilon^\alpha(E) = \sup_{\varepsilon>0} H_\varepsilon^\alpha(E),$$

where we take precise lower limit by all possible finite or countable covering of the set E by intervals E_j having diameters $d(E_j)$ less than ε.

Non-negative number α_0 such that

$$\alpha_0(A) = \sup\{\alpha : H^\alpha(E) = +\infty\} = \inf\{\alpha : H^\alpha(E) = 0\}$$

is called Hausdorff-Besicovitch or fractal dimension of the set E.

We study fractal dimension of the set of subsums for convergent positive series (2) depends on its initial parameters $k_1, k_2, \dots, k_m$ and q. In general case we can't calculate the fractal dimension of the set $E(a_n)$, however we can find some upper and lower limits for it. There are also some special cases when fractal dimension can be calculated precisely.

Example 1. The set of subsums for bigeometric series $(m = 2)$ having the form

$$k_1 + k_2 + k_1q + k_2q + k_1q^2 + k_2q^2 + \cdots + k_1q^{n-1} + k_2q^{n-1} + \dots,$$

where $k_1 \ge k_2$ are fixed positive integers and $q = k_2^2/k_1^2 < 1/cardA$, is a fractal set with the following Hausdorff-Besicovitch dimension

$$\dim_H E(a_n) = \frac{1}{\log_2 k_1 - \log_2 k_2}.$$

Example 2. The set of subsums for the series (2) having the property $k_1 = c^m$, $k_2 = c^{m-1}, \dots, k_{m-1} = c^2, k_m = c$ and $q = 1/c^m < 1/cardA$, $c \in N$, is a fractal set, moreover, it has the following Hausdorff-Besicovitch dimension

$$dim_H E(a_n) = \log_2^{-1} c.$$

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