On embedding of a locally nilpotent derivation on $K[x_1, \ldots, x_n]$ into $sl_2(K)$

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Let K be an algebraically closed field of characteristic zero, $K[x_1, \ldots, x_n]$ the ring of polynomials in n variables. Recall that a K-linear map $D\colon K[x_1, \ldots, x_n] \to K[x_1, \ldots, x_n]$ is called a K-derivation if $D(fg) = D(f)g + fD(g)$ for all $f, g \in K[x_1, \ldots, x_n]$. All K-derivations form a Lie algebra over the field K with respect to the Lie bracket given by $[D_1, D_2] = D_1D_2 - D_2D_1$. This Lie algebra is denoted by $W_n(K)$ and it was actively studied by many authors. From geometric viewpoint a derivation is a vector field with polynomial coefficients on the manifold K^n. Among all derivations locally nilpotent derivations are especially interesting for applications. Let us recall that a derivation $D \in W_n(K)$ is called locally nilpotent if for each $f \in K[x_1, \ldots, x_n]$ there exists a number $n = n(f)$ such that $D^n(f) = 0$. Many papers are devoted to studying locally nilpotent derivations, in particular the monography [3]. In theory of Lie algebras it is well known Morozov's Theorem, see, for instance, [2, Chapter III, Theorem 17]. It states that for a completely reducible linear Lie algebra its every nonzero nilpotent element can be embedded in its Lie subalgebra which is isomorphic to $\mathfrak{sl}_2(K)$. We prove an analogous result for locally nilpotent derivations.

Theorem 1. *Let L be a locally nilpotent derivation of the ring $K[x_1, \ldots, x_n]$. Then there exist derivations $\widehat{D}$ and H such that*

$$[D, \widehat{D}] = H, \quad [H, D] = 2D, \quad [H, \widehat{D}] = -2\widehat{D},$$

i.e. D is embeddable into a subalgebra isomorphic to $\mathfrak{sl}_2(K)$.

Note that for linear locally nilpotent derivations which are determined by a single Jordan block, such an embedding was used in the paper [1] for finding generators of the rings of constants of these derivations.

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