η-(SKEW) HERMITIAN REFLEXIVE SOLUTION TO A SYSTEM OF REAL QUATERNION MATRIX EQUATIONS

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Let $\mathbb{R}$ and $\mathbb{H}^{m\times n}$ stand respectively to the real number field and the set of all $m\times n$ matrices over the quaternion algebra

$$\mathbb{H}=\left\{a_0+a_1i+a_2j+a_3k \mid i^2=j^2=k^2=ijk=-1,\, a_0,\, a_1,\, a_2,\, a_3\in\mathbb{R}\right\}.$$

A matrix $A\in\mathbb{H}^{n\times n}$ is said to be a η-(skew) Hermitian if $A=A^{\eta*}$ $(A=-A^{\eta*})$ for $\eta\in\{i,j,k\}$. Motivated by [1, 2], in this paper we establish necessary and sufficient conditions for the existence of η-(skew) Hermitian reflexive solution of the system of quaternion matrix equations :

$$\begin{cases} AX=C \\ XB=D \\ NXN^{\eta*}=A_3 \end{cases} \tag{1}$$

and also provide the general expressions of solutions when this system has a solution. In addition, we give some results about the existence of the η- (skew) Hermitian reflexive solution of some classical linear systems.

Definition 1. Given a generalized reflection matrix $P\in\mathbb{H}^{n\times n}$ i.e. $P^*=P$ and $P^2=I_n$.
1. A matrix $A\in\mathbb{H}^{n\times n}$ is said to be an η- Hermitian reflexive matrix with respect to P if $A=A^{\eta*}$ and $A=PAP$.
2. A matrix $A\in\mathbb{H}^{n\times n}$ is said to be an η- skew-Hermitian reflexive matrix if $A=-A^{\eta*}$ and $A=PAP$.
We denote the set of all $n\times n$ η-Hermitian (η-skew-Hermitian) reflexive matrices with respect to P by $\hbar\mathbb{H}_r^{n\times n}(P)$ $(S\hbar\mathbb{H}_r^{n\times n}(P))$.

The main result of this work is the following.

Theorem 1. *Given a generalized reflection matrix $P\in\mathbb{H}^{n\times n}$. Let $A\in\mathbb{H}^{m\times n}$, $B\in\mathbb{H}^{n\times l}$, $C\in\mathbb{H}^{m\times n}$, $D\in\mathbb{H}^{n\times l}$, $N\in\mathbb{H}^{q\times n}$ and $A_3=A_3^{\eta*}\in\mathbb{H}^{q\times q}$ for ($\eta\in\{i,j,k\}$) be given. We put*

$$E=\begin{bmatrix} A_1 \\ B_1^{\eta*}\end{bmatrix},\ F=\begin{bmatrix} C_1 \\ D_1^{\eta*}\end{bmatrix},\ G=\begin{bmatrix} A_2 \\ B_2^{\eta*}\end{bmatrix},\ H=\begin{bmatrix} C_2 \\ D_2^{\eta*}\end{bmatrix},$$
$$K=C_3L_E,\ J=D_3L_G,\ M=R_KJ,\ S=JL_M,$$

$$\begin{aligned}Q=&A_3-C_3E^+FC_3^{\eta*}-C_3F^{\eta*}\left(E^+\right)^{\eta*}C_3^{\eta*}-D_3G^+HD_3^{\eta*}-D_3H^{\eta*}\left(G^+\right)^{\eta*}D_3^{\eta*}\\&+C_3E^+EF^{\eta*}\left(E^+\right)^{\eta*}C_3^{\eta*}+D_3G^+GH^{\eta*}\left(G^+\right)^{\eta*}D_3^{\eta*}.\end{aligned}$$

Then, the following statements are equivalent:
1) The system (1) has a η- Hermitian reflexive solution $X\in\hbar\mathbb{H}_r^{n\times n}(P)$.
2)

$$\begin{aligned}&EF^{\eta*}=FE^{\eta*},\\&GH^{\eta*}=HG^{\eta*},\\&R_EF=R_GH=R_MR_KQ=R_KQ(R_J)^{\eta}=0.\end{aligned}$$

3)

$$r\begin{bmatrix} A_1 & C_1 \\ B_1^{\eta*} & D_1^{\eta*} \end{bmatrix} = r\begin{bmatrix} A_1 \\ B_1^{\eta*} \end{bmatrix},$$

$$r\begin{bmatrix} A_2 & C_2 \\ B_2^{\eta*} & D_2^{\eta*} \end{bmatrix} = r\begin{bmatrix} A_2 \\ B_2^{\eta*} \end{bmatrix},$$

$$r\begin{bmatrix} A_3 & D_3 & C_3 \\ C_2 D_3^{\eta*} & A_2 & 0 \\ D_2^{\eta*} D_3^{\eta*} & B_2^{\eta*} & 0 \\ C_1 C_3^{\eta*} & 0 & A_1 \\ D_1^{\eta*} C_3^{\eta*} & 0 & B_1^{\eta*} \end{bmatrix} = r\begin{bmatrix} D_3 & C_3 \\ A_2 & 0 \\ B_2^{\eta*} & 0 \\ 0 & A_1 \\ 0 & B_1^{\eta*} \end{bmatrix},$$

$$r\begin{bmatrix} A_3 & C_3 & D_3 C_2^{\eta*} & D_3 D_2 \\ D_3^{\eta*} & 0 & A_2^{\eta*} & B_2 \\ C_1 C_3^{\eta*} & A_1 & 0 & 0 \\ D_1^{\eta*} C_3^{\eta*} & B_1^{\eta*} & 0 & 0 \end{bmatrix} = r\begin{bmatrix} C_3 \\ A_1 \\ B_1^{\eta*} \end{bmatrix} + r\begin{bmatrix} D_3 \\ A_2 \\ B_2^{\eta*} \end{bmatrix}.$$

In this case the η- Hermitian reflexive solution of the system (I) can be expressed as the following

$$X = U\begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} U^{\eta*} \in \hbar\mathbb{H}_r^{n\times n}(P)$$

where

$$\begin{aligned} X_1 = X_1^{\eta*} &= E^+F + F^{\eta*}(E^+)^{\eta*} - E^+EF^{\eta*}(E^+)^{\eta*} + L_E K^+ Q (K^+)^{\eta*}(L_E)^{\eta*} \\ &- L_E K^+ S W_2 (L_E K^+ S)^{\eta*} - \frac{1}{2} L_E K^+ J M^+ Q[I + (SJ^+)^{\eta*}](L_E K^+)^{\eta*} \\ &- \frac{1}{2} L_E K^+ (I + SJ^+) Q (L_E K^+ J M^+)^{\eta*} + L_E L_K V_1^{\eta*}(L_E)^{\eta*} + L_E V_1 (L_E L_K)^{\eta*}, \end{aligned} \tag{2}$$

$$\begin{aligned} X_2 = X_2^{\eta*} &= G^+H + H^{\eta*}(G^+)^{\eta*} - G^+GH^{\eta*}(G^+)^{\eta*} + \frac{1}{2} L_G M^+ Q (J^+)^{\eta*}[I + (S^+S)^{\eta*}](L_G)^{\eta*} \\ &+ \frac{1}{2} L_G (I + S^+S) J^+ Q (L_G M^+)^{\eta*} + L_G L_M W_2 (L_G L_M)^{\eta*} + L_G L_M L_S W_1 (L_G)^{\eta*} + \\ &L_G W_1^{\eta*} (L_G L_M L_S)^{\eta*} + L_G L_J V_2^{\eta*} (L_G)^{\eta*} + L_G V_2 (L_G L_J)^{\eta*}. \end{aligned} \tag{3}$$

where V_1, V_2, W_1 and $W_2 = W_2^{\eta*}$ are arbitrary matrices over $\mathbb{H}$ with appropriate sizes.

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