Line digraphs of polytrees and their weak components

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Given a digraph $D = (V(D), A(D))$, its line digraph is a digraph $L(D)$ with the vertex set $V(L(D)) = A(D)$ and the arc set

$$A(L(D)) = \{(\alpha, \beta) \in A(D) \times A(D) : \alpha = (a, b), \beta = (b, c) \text{ for } a, b, c \in V(D)\}.$$

A bipartite digraph is an orientation of a bipartite graph in which all arcs are oriented from one part (of the corresponding bipartition) to another. A two-port digraph is a bipartite digraph D with $[D]$ being a complete bipartite graph. We denote it by $K(A, B)$ so that every arc of the digraph starts with a vertex from set A and ends with a vertex from set B.

The most important characterization of line digraphs is due to Harary and Norman [2].

Theorem 1. [2] *A digraph D is a line digraph if and only if there exist two improper partitions A_i and B_i of $V(D)$ into n subsets such that $L(D) = \cup K(A_i, B_i)$.*

More on line digraphs can be found in a recent survey paper [1].

A polytree, polypath, polystar, polycycle is an orientation of a tree, path, star, cycle, respectively. An in-tree (out-tree) is a polytree in which all edges are oriented towards (from) some sink (source).

Our first result is a characterization of line digraphs of polytrees, as well as certain subclasses of polytrees.

Theorem 2. *Let D be a line digraph. Then there exists a polytree T with $L(T) \simeq D$ if and only if every polycycle in D is a bipartite subgraph.*

Proposition 1. *Let D be a digraph. Then $D \simeq L(T)$ for an in-tree (out-tree) T if and only if every weak component of D is an in-tree (out-tree).*

Proposition 2. *Let D be a digraph. Then $D \simeq L(T)$ for a polypath T if and only if every weak component of D is a dipath.*

Proposition 3. *Let D be a digraph. Then $D \simeq L(S)$ for a polystar S if and only if D is either a two-port digraph or an empty digraph.*

Next we count the number of weak components in the line digraph of a polytree. Denote by $Si(D)$ and $So(D)$ the sets of sinks and sources in D, respectively.

Proposition 4. *Let T be a polytree with $n \geq 2$. Then the number of weak components in $L(T)$ equals*

$$\sum_{u \in Si(T) \cup So(T)} (d_{[T]}(u) - 1) + 1.$$

Corollary 1. *Let X be a tree. Then the average number of weak components in $L(T)$ among orientations T of X equals*

$$\sum_{u \in V(X)} \frac{d_X(u) - 1}{2^{d_X(u)-1}} + 1.$$

We also provide an algorithm that for a given polytree T constructs its induced subgraphs that become weak components in $L(T)$.

Algorithm 1. ***Require:*** *Polytree T*

Ensure: *Set C of induced subgraphs of T which are mapped to different weak components when applying line digraph operator*

1: $C \leftarrow \emptyset$.
2: $S_{out} \leftarrow$ *the set of yet unvisited out-arcs from sources of* T.
3: ***while*** $S_{out} \neq \emptyset$ ***do***
4:     $s \leftarrow$ *an out-arc from* S_{out}.
5:     $f \leftarrow$ *an in-arc to sink reachable from* s.
6:     $P \leftarrow$ *the dipath from* s *to* f.
7:     $P_{prev} \leftarrow$ *empty digraph.*
8:     ***while*** $P_{prev} \neq P$ ***do***
9:         $P_{prev} \leftarrow P$.
10:         $P \leftarrow T[\{\alpha \in A(T) | \alpha \text{ is connected with some arc from } P\}]$.
11:     ***end while***
12:     $C \leftarrow C \oplus P$.
13:     $S_{out} \leftarrow S_{out} \backslash A(P)$.
14: ***end while***

Remark 1. After every iteration of the **while** cycle (lines $8 - 11$) we either have added new arcs to P, or $P = P_{prev}$ and we terminate. Since the total number of arcs in T is finite, this process cannot continue indefinitely.
As for the main **while** cycle (lines $3 - 14$), nonemptiness of S_{out} implies the existence of s. Since T is finite and contains no directed cycle, f also exists. Then P as obtained in line 6 is not empty. Thus in line 12 we add a nonempty subgraph of T to C. Since, the total number of nonempty subgraphs is finite, the algorithm terminates.

The correctness of the algorithm is justified in the following theorem.

Theorem 3. *Let $T_1, T_2, \ldots, T_n$ be subgraphs obtained by applying Algorithm 1 to polytree T. Then the next statements hold:*

1. *T_k is a weak subgraph for all $k \in [1, n]$.*
2. *$A(T_1) \sqcup A(T_2) \sqcup \cdots \sqcup A(T_n) = A(T)$.*
3. *$L(T_k)$ is a weak graph for all $k \in [1, n]$.*
4. *$L(T_k \cup \{\alpha\})$ is not weak, for all $k \in [1, n]$ and for every $\alpha \in A(T) \backslash A(T_k)$.*

1. Bagga J. S., Beineke L. W., A survey of line digraphs and generalizations. Discrete Math. Lett., 2021, 6, 68–93.
2. Harary F., Norman R. Z. Some Properties of Line Digraphs. Rend. Circ. Mat. Palermo, 1960, 9, 161–168.