On some arithmetic functions for Gaussian integers

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Gaussian integers are complex numbers whose real and imaginary parts are integers. In this research, we study the properties of some number-theoretic functions—the number of divisors $\tau^*(\alpha)$, sum of the m-th powers of the divisors $\sigma^*_m(\alpha)$ and product of the divisors $\pi^*(\alpha)$ of a non-zero Gaussian integer α—that are defined as follows:

$$\tau^*(\alpha) = \sum_{\delta|\alpha} 1, \quad \sigma^*_m(\alpha) = \sum_{\delta|\alpha} \delta^m, \quad \pi^*(\alpha) = \prod_{\delta|\alpha} \delta.$$

For a non-zero Gaussian integer α that is not a unit and has a prime factorization of the form $\mu\rho_1^{a_1}\rho_2^{a_2}\cdot\ldots\cdot\rho_k^{a_k}$, we obtained the following calculation formulae of the above-mentioned functions:

$$\tau^*(\alpha) = 4\prod_{j=1}^{k}(a_j+1), \quad \sigma^*_{4m}(\alpha) = 4\prod_{j=1}^{k}\frac{\rho_j^{4m(a_j+1)}-1}{\rho_j^{4m}-1},$$

$$\pi^*(\alpha) = \begin{cases} \alpha^{\frac{1}{2}\tau^*(\alpha)} & \text{if } \alpha \text{ is not a square of a Gaussian integer;} \\ -\alpha^{\frac{1}{2}\tau^*(\alpha)} & \text{if } \alpha \text{ is a square of a Gaussian integer.} \end{cases}$$

The criteria of divisibility of the number and sum of the divisors by certain numbers were proven. In particular, we showed that $\tau^*(\alpha)$ is divisible by 8 if and only if α is not an associate of a square of a non-zero Gaussian integer; $\sigma^*_{4m}(\alpha), m \in \mathbb{N}$ is divisible by 8 if and only if neither α nor $(1+i)\alpha$ is an associate of a square of a Gaussian integer; and that otherwise $\sigma^*_{4m}(\alpha)$ is not divisible by $4(1+i)$.

The values of the τ^* and $\sigma^*_{4m}, m \in \mathbb{N}$, functions for a non-zero Gaussian integer α were estimated from above with its absolute value. For the σ^*_{4m} function, we also established the lower and upper bounds that use the radical $\operatorname{rad}^*(\alpha)$ of α—the product of its prime divisors:

$$\frac{4|\alpha|^{4m}}{|\operatorname{rad}^*(\alpha)|^{4m\log_4\frac{4}{3}}} \le |\sigma^*_{4m}(\alpha)| \le 4|\alpha|^{4m}|\operatorname{rad}^*(\alpha)|^{4m\log_4\frac{17}{12}}.$$

Aside from that, it was proven that if $\beta|\alpha$, where $\alpha, \beta \in \mathbb{Z}[i] \setminus \{0\}$, then the number, sum of the $4m$-th powers and product of those divisors of α that are divisible by β equal $\tau^*(\frac{\alpha}{\beta}), \beta^{4m}\sigma^*_{4m}(\frac{\alpha}{\beta})$ and $\beta^{\tau^*(\frac{\alpha}{\beta})}\pi^*(\frac{\alpha}{\beta})$, respectively.

The other objects studied are sums of products containing a fixed number of divisors of a Gaussian integer; and sums of their reciprocals:

$$s^*_{k,m}(\alpha) = \sum_{1\le a_1<a_2<\ldots<a_k\le\tau^*(\alpha)} \delta^m_{a_1}\delta^m_{a_2}\cdot\ldots\cdot\delta^m_{a_k};$$

$$p^*_{k,m}(\alpha) = \sum_{1\le a_1<a_2<\ldots<a_k\le\tau^*(\alpha)} \frac{1}{\delta^m_{a_1}\delta^m_{a_2}\cdot\ldots\cdot\delta^m_{a_k}}.$$

For these sums, we obtained the calculation formulae for the case of two and three divisors; the transition formulae connecting values of the $s^*_{k,m}$ and $p^*_{k,m}$ functions; and the recurrence formulae allowing one to calculate sums of products of any number of divisors:

$$s^*_{k+1,m}(\alpha) = \frac{1}{k+1}\left((-1)^k \sigma^*_{(k+1)m}(\alpha) + \sum_{l=0}^{k-1} (-1)^l s^*_{k-l,m}(\alpha)\sigma^*_{(l+1)m}(\alpha)\right);$$

$$p^*_{k+1,m}(\alpha) = \frac{1}{k+1}\left(\frac{(-1)^k \sigma^*_{(k+1)m}(\alpha)}{\alpha^{(k+1)m}} + \sum_{l=0}^{k-1} \frac{(-1)^l p^*_{k-l,m}(\alpha)\sigma^*_{(l+1)m}(\alpha)}{\alpha^{(l+1)m}}\right).$$

Among the obtained properties, there are also some conditions when these sums take either integer or rational values; or when they are equal to zero. We also studied values of the analogous sums for those divisors of a Gaussian integer that are divisible by another Gaussian integer.

The practical value of the outcomes of this research was shown in the compact solutions to the self-created problems that were constructed based on the formulated propositions.

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