Graphs with parity conditions between non-cut vertices

K. O. Antoshyna[1], S. O. Kozerenko[2]
[1]Kyiv Academic University, Kyiv, Ukraine
[2]National university of Kyiv-Mohyla Academy, Kyiv, Ukraine
k.antoshyna@kau.edu.ua, sergiy.kozerenko@ukma.edu.ua

In this work all the graphs are finite and connected. On the vertex set of a graph G the metric d_G is naturally defined: for each pair $u, v \in V(G)$ the distance $d_G(u, v)$ equals the length of a shortest path between u and v. A vertex $u \in V(G)$ is called a *cut vertex*, if $G - u$ is disconnected (see [1, p. 26]).

A graph G is called an *NCE-graph*, if for each pair u, v of the cut vertices in G the distance $d_G(u, v)$ is even.

Theorem 1. *Every NCE-graph is bipartite.*

From this theorem one can derive the following criterion of the NCE-graphs. Recall that for connected bipartite graphs the corresponding bipartition is unique (up to switching the parts).

Corollary 1. *The graph G is an NCE-graph if and only if G is bipartite having all its non-cut vertices in a common part of the corresponding bipartition.*

It turns out that any bipartite graph can be embedded into an NCE-graph.

Proposition 1. *Any bipartite graph is an induced subgraph of an NCE-graph.*

Recall that a vertex set $A \subset V(G)$ of a graph G is called *independent*, if no vertices of A are adjacent in G.

Proposition 2. *A graph G with $|V(G)| \geqslant 3$ can be subdivided to an NCE-graph if and only if the set of non-cut vertices in G is independent.*

A graph G is called an *NCO-graph*, if for each pair u, v of the cut vertices in G the distance $d_G(u, v)$ is odd. To present the criterion of NCO-graphs, we need the following standard construction. Let G be a graph. Its *line graph* $L(G)$ is the intersection graph for the family $E(G)$. In other words, the vertices in $L(G)$ are the edges in G with two edges being adjacent in $L(G)$ provided they share a common vertex in G. Using the characterization of line graphs of trees (see [2]), we obtained the following criterion for NCO-graphs.

Theorem 2. *A connected graph is an NCO-graph if and only if it is a line graph of an NCE-tree.*

Similarly to the NCE-graphs, for NCO-graphs we obtain characterizations of their induced subgraphs and graphs, which can be subdivided to them.

Proposition 3. *A graph is isomorphic to an induced subgraph of an NCO-graph if and only if it is a line graph of a tree.*

Proposition 4. *A graph can be subdivided to an NCO-graph if and only if it is a line graph of a tree, which is a subdivision of some NCE-tree.*

1. Harary F. Graph theory. — Mass.: Addison-Wesley, Reading, 1969, 274.
2. Harary F. A characterization of block graphs. Canad. Math. Bull., 1963, 6 , 1–6.