Apollonius Problem and Caustics of an Ellipsoid

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In the talk we will discuss Apollonius Problem on the number of normals of an ellipse passing through a given point. By following the footsteps of Apollonius, it can be shown that the number is dependent on the position of the given point with respect to a certain astroida. The special case when the point is on the ellipse is studied using the intersection points of the astroida and the ellipse. The problem is then generalized for 3 dimensional space, namely for Ellipsoids. The number in this case is shown to be dependent on the position of the given point with respect to caustics of the ellipsoid. If the given point is on the ellipsoid then the number of normals is dependent on position of the point with respect to the intersections of the ellipsoid with its caustics.

How many normals can one draw from a point to an ellipse? In the current talk we will try to solve this problem and its generalization to 3 dimensions, using the modern methods of mathematics, which were not around when Apollonius of Perga (c. III-II centuries BC) first asked and answered this question in his famous work *Conics*.

Let the ellipse be defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{1}$$

where we assume that $a > b > 0$. Let us take an arbitrary point $A(X, Y)$ on the plane of the ellipse. We can find point $B(x, y)$ on the ellipse such that AB is perpendicular to the tangent of the ellipse at B. Let us denote by $n(A)$ the total number of intersections of the hyperbola with the ellipse. Since the intersection points are the solutions of a fourth order equation, $n(A)$ can not exceed 4. Let us find points A, where $n(A)$ jumps from 4 to 2:

$$\sqrt[3]{a^2X^2} + \sqrt[3]{b^2Y^2} = \sqrt[3]{(a^2 - b^2)^2}. \tag{2}$$

It is the equation of *astroida* in X, Y coordinates. In the interior region of the astroida $n(A) = 4$. Outside of the astroida $n(A) = 2$. On the astroida itself $n(A) = 3$, except the vertices of the astroida $\left(\pm\frac{a^2-b^2}{a}, 0\right)$ and $\left(0, \pm\frac{a^2-b^2}{b}\right)$, where again $n(A) = 2$. This is essentially what was done by Apollonius, which is a remarkable achievement, taking into account the mathematical tools available at the time. It is well known that this astroida is the evolute of the ellipse and therefore drawing normals to the ellipse can be done by drawing tangent lines of the astroida.

Let us now suppose that the point $A(X, Y)$ is on the ellipse: $X = x$, $Y = y$. Since Apollonius hyperbola passes through $A(X, Y)$, automatically, one of the normals disappears, because one of the points B_1, B_2, B_3, and B_4, coincide with A. For the points of the ellipse (1) in the astroida (2), $n(A) = 3$. For the points of the ellipse (1) outside the astroida (2), $n(A) = 1$. For the intersection points N_1, N_2, N_3, and N_4 of the ellipse (1) and the astroida (2), $n(A) = 2$. The coordinates of these points can be easily determined: $(\pm x_0, \pm y_0)$ and $(\pm x_0, \mp y_0)$, where

$$x_0 = \sqrt{\frac{a^4(a^2 - 2b^2)^3}{(a^2 - b^2)(a^2 + b^2)^3}},\ y_0 = \sqrt{\frac{b^4(2a^2 - b^2)^3}{(a^2 - b^2)(a^2 + b^2)^3}}.$$

Thus we proved the following

Theorem 1. *For the ellipse (1) and the astroida (2), the following cases are possible:*

1. *If $a^2 > 2b^2$ then the points $(\pm x_0, \pm y_0)$ and $(\pm x_0, \mp y_0)$ separate the ellipse into 4 regions where $n(A) = 3$ and $n(A) = 1$.*
2. *If $a^2 \leq 2b^2$, then for all the points of the ellipse (1), $n(A) = 1$.*

Noting this, we can say that Apollonius problem for the number of concurrent normals of an ellipse is completely solved. Let us now consider three dimensional generalization of this problem. How many concurrent normals of an ellipsoid are there? Let an ellipsoid be defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \tag{3}$$

where we assume that $a > b > c > 0$. Let us take an arbitrary point $A(X, Y, Z)$ and find the number $n(A)$ of points $B(x, y, z)$ on the ellipsoid such that AB is the normal line of the plane tangent to the ellipsoid at B. As before, let us denote the number of normals through A by $n(A)$. We want to find points A, where $n(A)$ jumps from 2 to 4, or from 4 to 6:

$$\left(\frac{aX}{a^2+t}\right)^2 + \left(\frac{bY}{b^2+t}\right)^2 + \left(\frac{cZ}{c^2+t}\right)^2 = 1, \tag{4}$$

$$\frac{a^2X^2}{(a^2+t)^3} + \frac{b^2Y^2}{(b^2+t)^3} + \frac{c^2Z^2}{(c^2+t)^3} = 0. \tag{5}$$

The equations (4) and (5) define the surface known as *Caustics of an Ellipsoid* also known as *focal surface, surface of centers, evolute of an ellipsoid* or *Cayley's astroida* [1]. Cayley used the name *Centro-surface of an Ellipsoid*. The number of normals outside of the two caustics is 2 ($n(A) = 2$). For the points of the space inside of only one and both of the caustics, $n(A) = 4$ and $n(A) = 6$, respectively. On the caustics $n(A) = 3$ or $n(A) = 5$, with some exceptions on the planes $X = 0$, $Y = 0$, and $Z = 0$ and on the intersections of the two caustics, where again $n(A) = 2$ or $n(A) = 4$. As for an ellipse, if the point A is on the ellipsoid (3) then the number of normals $n(A)$ through A decreases by 1. The following theorem is proved.

Theorem 2. *For the ellipsoid (3) and its caustics defined by (4) and (5), the following cases are possible:*

1. *If $a^2 < 2c^2$ then there are no intersections of the caustics with the ellipsoid (3),*
2. *If $b^2 < 2c^2 \leq a^2$ then only one of the caustics intersects the ellipsoid (3),*
3. *If $b^2 \geq 2c^2$, then both of the caustics intersects the ellipsoid (3).*

In all cases, for the points of the ellipsoid (3) lying outside of the two caustics $n(A) = 1$, for the points of the ellipsoid (3) lying in only one of these caustics $n(A) = 3$, for the points of the ellipsoid (3) lying in both of these caustics $n(A) = 5$, and for the intersection points of the ellipsoid (3) and these caustics $n(A) = 2$ or 4, except some of the points of the ellipsoid (3), where the caustics intersect each other or these caustics intersect the coordinate planes.

More detailed categorization of the cases of intersection of the ellipsoid and its caustics can be done based on the sign of expressions $\frac{1}{a^2} + \frac{1}{c^2} - \frac{3}{b^2}$, $a^2 + c^2 - 2b^2$ and other similar expressions.

1. Cayley A. On the centro-surface of an ellipsoid. Transactions of the Cambridge Philosophical Society., 1873, 12, 1, 319-365. Also included in The collected mathematical papers of Arthur Cayley, Vol. VIII, Cambridge University Press, Cambridge, 316-365, 1895. `http://name.umdl.umich.edu/ABS3153.0008.001`