On the solution of separate differential equations with variational derivatives of the first and second orders

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The theory of equations with variational (functional) derivatives is a fairly extensive area of mathematics. This class of equations has numerous applications in statistical physics, quantum field theory, hydromechanics and other fields. The theory of variational derivatives, differential and integro-differential equations with variational derivatives is quite fully stated, for example, in the monographs [1–3] and the works [4–10].

The problems that are formulated and investigated in this area of mathematics are similar to the problems considered in the case of ordinary and partial differential equations. Explicit formulas for solving equations with variational derivatives are known only in a few cases. This applies mainly to the set of linear equations [5–8]. Therefore, the main methods for solving such equations are approximate.

The problem of an approximate solution of equations with variational derivatives is not sufficiently studied. When solving this class of problems, it may be useful to apply methods that take into account the given initial and boundary values. In particular, in the Cauchy problem for an equation of the n-th order, the desired functional $F(x)$ can be approximately found from the values of the functional $F(x_0)$ and its variational derivatives up to the $(n-1)$-th order, which are known at the point $x_0(t)$. For this, it is natural to use the operator interpolation apparatus [11, 12]. Consider one of the ways for the approximate solution of equations with variational derivatives, based on interpolation of the functional included in the equation.

We formulate the definition of the variational derivative for functionals defined on sets of functions [9]. Let X be a linear space of real functions defined on a segment $[a, b]$ of the real axis $\mathbb{R}$, and F be an operator or functional given on the X.

The k-th order Gateaux differential $\delta^k F[x; h_1, h_2, ..., h_k]$ $(k \in \mathbb{N})$ of the mapping F at the point $x \in X$ in the directions $h_1, h_2, ..., h_k \in X$ is defined by the equality

$$\delta^k F[x; h_1, h_2, ..., h_k] =$$

$$= \lim_{\lambda \to 0} \frac{\delta^{k-1} F[x + \lambda h_k; h_1, h_2, ..., h_{k-1}] - \delta^{k-1} F[x; h_1, h_2, ..., h_{k-1}]}{\lambda} =$$

$$= \left. \frac{\partial^k F\left(x + \lambda_1 h_1 + \lambda_2 h_2 + ... + \lambda_k h_k\right)}{\partial \lambda_k \cdots \partial \lambda_1} \right|_{\lambda_1 = ... = \lambda_k = 0}, \delta^0 F[x] \equiv F\left(x\right).$$

If there exists the k-th order Gateaux differential $\delta^k F[x; h_1, h_2, ..., h_k]$ $(x, h_i \in X;\; i = 1, 2, ..., k)$ of the functional $F(x)$ at the point $x \in X$ in the directions $h_1, h_2, ..., h_k \in X$, that can be represented as

$$\delta^k F[x; h_1, h_2, ..., h_k] = \int_{[a,b]^{\,k}} a\left(x; t_1, ..., t_k\right) h_1\left(t_1\right) ... h_k\left(t_k\right) dt_1 ... dt_k, \qquad (1)$$

where $a\left(x; t_1, ..., t_k\right)$ is some function depending on $x = x\left(s\right)$ and variables $t_1, ..., t_k \in \mathbb{R}$, then $a\left(x; t_1, ..., t_k\right)$ is called the variational derivative of the k-th order of the functional $F(x)$ with respect to x at the point $t = (t_1, t_2, ..., t_k)$ and denoted by the symbol $\dfrac{\delta^k F(x)}{\delta x(t_1) \cdots \delta x(t_k)}$.

As X, one can choose the space $C[a, b]$ of continuous functions with a uniform norm, the Hilbert space $L_2[a, b]$ or any other space such that the integral on the right-hand side of (1) makes sense.

We give formulas for the exact solution of some of the simplest differential equations with variational derivatives.

For example, for the equation

$$\frac{\delta F(x)}{\delta x(t)} = 2p\left(t\right) \cos x\left(t\right) \int_0^1 p\left(t\right) \sin x\left(t\right) dt,$$

the solution is the functional $F\left(x\right) = \left(\int_0^1 p\left(t\right) \sin x\left(t\right) dt\right)^2$, where $p\left(t\right)$ and $x\left(t\right)$ are elements of space $C[0, 1]$.

The functional $F\left(x\right) = \int_a^b p\left(t\right) f\left[x\left(t\right)\right] dt$ is the solution of the equation $\dfrac{\delta F(x)}{\delta x(t)} = p\left(t\right) f'\left[x\left(t\right)\right]$.

The solution of the equation

$$\frac{\delta F(x)}{\delta x(t)} = p_0\left(t\right) e^{x(t)} + p_1(t) \cos x\left(t\right) + p_2\left(t\right) \sin x\left(t\right) + p_3\left(t\right) x\left(t\right) +$$

$$+ p_4\left(t\right) \int_0^1 p_4\left(\tau\right) x\left(\tau\right) d\tau$$

has the form

$$F\left(x\right) = \int_0^1 \left[p_0\left(t\right) e^{x(t)} + p_1\left(t\right) \sin x\left(t\right) - p_2(t) \cos x\left(t\right) + \frac{1}{2} \left(\int_0^1 p_4\left(t\right) x\left(t\right) dt\right)^2\right] p\left(t\right) \sin x\left(t\right) dt,$$

where $p_0\left(t\right)$ $(i = 0, 1, ..., 4)$ are arbitrary functions for which reduced integrals exist.

Next, we consider a differential equation of the hyperbolic type with the second-order variational derivatives:

$$\frac{\delta^2 u(x,y)}{\delta x^2(t)} - a^2(t)\frac{\delta^2 u(x,y)}{\delta y^2(t)} = 0$$

$$\left(x = x(t) \geq 0,\ y = y(t),\ a(t) \neq 0;\ \ t \in [a,b] \subseteq \mathbb{R}\right). \tag{2}$$

The solution of this equation is the functional

$$u(x,y) = f_1\left[\int_a^b (y(t) + a(t)\,x(t))\,dt\right] + f_2\left[\int_a^b (y(t) - a(t)\,x(t))\,dt\right], \tag{3}$$

where $f_1(\cdot)$ è $f_2(\cdot)$ are any functions that are twice differentiable on $\mathbb{R}$. The representation (3) is an analogue of the classical Dalamber formula.

We give the Hermite interpolation formula $H(x,y)$ with respect to a single node of the second multiplicity, which is an approximation to the solution $u(x,y)$ of the Cauchy problem for equation (2) with the initial conditions

$$u(x_0,y) = u_0(y),\ \frac{\delta u(x_0,y)}{\delta x} = u_1(y), \tag{4}$$

where $u_0(y)$ and $u_1(y)$ are some functionals defined on $C[a,b]$.

Theorem 1. *An approximate solution of the Cauchy problem (2), (4) can be represented as*

$$H(x,y) = u_0(y) + u_1(y)\int_a^b (x(t) - x_0(t))\,dt + \frac{1}{2}a^2(t)\,u_0''(y)\,[x(t) - x_0(t)]^2. \tag{5}$$

The proof of this theorem is based on a direct verification of the interpolation conditions (4).

Substituting the approximation $H(x,y)$ of the form (5) to the solution $u(x,y)$ of the equation (2) in the left-hand side of equality (2), we obtain

$$\frac{\delta^2 H(x,y)}{\delta x^2(t)} - a^2(t)\frac{\delta^2 H(x,y)}{\delta y^2(t)} = -a^2(t)\cdot$$

$$\left(\int_a^b (x(t) - x_0(t))\,dt\,u_1''(y) + \frac{1}{2}a^2(t)\,u_0^{(4)}(y)\,[x(t) - x_0(t)]^2\right)\delta(s-t)\,\delta(s_1-t),$$

where the delta function $\delta(t) = \begin{cases} 0, & t \neq 0; \\ +\infty, & t = 0. \end{cases}$ In particular, at the point (x_0,y) we have

$$\frac{\delta^2 H(x_0,y)}{\delta x^2(t)} - a^2(t)\frac{\delta^2 H(x_0,y)}{\delta y^2(t)} = 0.$$

We note that in the case $u_1''(y) = u_0^{(4)}(y) \equiv 0$, the equality

$$\frac{\delta^2 H(x,y)}{\delta x^2(t)} - a^2(t)\frac{\delta^2 H(x,y)}{\delta y^2(t)} = 0$$

takes place for any (x,y) from the domain of definition.

The obtained results can serve as a basis for further research of the theory of differential equations with variational derivatives that is not well developed, and can also be used to construct approximate interpolation methods for solving some linear and nonlinear differential equations with variational derivatives of the first and second order that are found in various applied fields and mathematical physics.

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