On global behavior of mappings in terms of prime ends

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An *end* of a domain D is an equivalence class of chains of cross-cuts of D. We say that an end K is a *prime end* if K contains a chain of cross-cuts $\{\sigma_m\}$, such that $\lim\limits_{m\rightarrow\infty} M(\Gamma(C, \sigma_m, D)) = 0$ for some continuum C in D, where M is the modulus of the family $\Gamma(C, \sigma_m, D)$. We say that the boundary of a domain D in $\mathbb{R}^n$ is *locally quasiconformal* if every point $x_0 \in \partial D$ has a neighborhood U that admit a conformal mapping φ onto the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$ such that $\varphi(\partial D \cap U)$ is the intersection of $\mathbb{B}^n$ and a coordinate hyperplane. We say that a bounded domain D in $\mathbb{R}^n$ is *regular* if D can be mapped quasiconformally onto a bounded domain with a locally quasiconformal boundary. If $\overline{D}_P$ is the completion of a regular domain D by its prime ends and g_0 is a quasiconformal mapping of a domain D_0 with locally quasiconformal boundary onto D, then this mapping naturally determines the metric $\rho_0(p_1, p_2) = \left|\widetilde{g_0}^{-1}(p_1) - \widetilde{g_0}^{-1}(p_2)\right|$, where $\widetilde{g_0}$ is the extension of g_0 onto $\overline{D_0}$. In what follows, given $p \geqslant 1$, M_p denotes the n-modulus of a family of paths, and the element $dm(x)$ corresponds to a Lebesgue measure in $\mathbb{R}^n$, $n \geqslant 2$. For given sets E and F and a given domain D in $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$, we denote by $\Gamma(E, F, D)$ the family of all paths $\gamma : [0, 1] \rightarrow \overline{\mathbb{R}^n}$ joining E and F in D, that is, $\gamma(0) \in E$, $\gamma(1) \in F$ and $\gamma(t) \in D$ for all $t \in [0, 1]$. Let $x_0 \in \overline{D}$, $x_0 \neq \infty$, $S(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}$, $A = A(x_0, r_1, r_2) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}$. Let $Q : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a Lebesgue measurable function satisfying the condition $Q(x) \equiv 0$ for $x \in \mathbb{R}^n \setminus D$. A mapping $f : D \rightarrow \overline{\mathbb{R}^n}$ is called a *ring Q-mapping at the point $x_0 \in \overline{D} \setminus \{\infty\}$ with respect to p-modulus*, if the condition

$$M_p(f(\Gamma(S(x_0, r_1), S(x_0, r_2), D))) \leqslant \int\limits_{A\cap D} Q(x) \cdot \eta^p(|x - x_0|)\, dm(x) \qquad (1)$$

holds for all $0 < r_1 < r_2 < d_0 := \sup\limits_{x\in D} |x - x_0|$ and all Lebesgue measurable functions $\eta : (r_1, r_2) \rightarrow [0, \infty]$ such that $\int\limits_{r_1}^{r_2} \eta(r)\, dr \geqslant 1$. Set $\omega_{n-1} = \mathcal{H}^{n-1}(S(0, 1))$, $q_{x_0}(r) := \frac{1}{\omega_{n-1}r^{n-1}} \int\limits_{|x-x_0|=r} Q(x)\, d\mathcal{H}^{n-1}$. In what follows, h is a chordal (spherical) metric in $\overline{\mathbb{R}^n}$. Given $p \geqslant 1$, $\delta > 0$, a domain $D \subset \mathbb{R}^n$,

$n \geqslant 2$, a continuum $A \subset D$ and a Lebesgue measurable function $Q : D \to [1, \infty]$ we denote $\mathfrak{F}_{Q,A,p,\delta}(D)$ the family of all ring Q-homeomorphisms $f : D \to \overline{\mathbb{R}^n}$ in $\overline{D}$ with respect to p-modulus satisfying the conditions $h(f(A)) \geqslant \delta$ and $h(\overline{\mathbb{R}^n} \setminus f(D)) \geqslant \delta$.

Theorem 1. *Let $p \in (n-1, n]$, let D be regular domain and let $D_f' = f(D)$ be equi-uniform family of bounded domains with respect to p-modulus over all $f \in \mathfrak{F}_{Q,A,p,\delta}(D)$, while $f(D)$ have locally quasiconformal boundaries. If Q either has a finite mean oscillation in $\overline{D}$, or the condition $\int\limits_0^{\beta(x_0)} \frac{dt}{t^{\frac{n-1}{p-1}} q_{x_0}^{\frac{1}{p-1}}(t)} = \infty$, holds for some $\beta(x_0) > 0$ and every point $x_0 \in \overline{D}$, then each $f \in \mathfrak{F}_{Q,A,p,\delta}(D)$ has a continuous extension $\overline{f} : \overline{D}_P \to \overline{\mathbb{R}^n}$, and the family $\mathfrak{F}_{Q,A,p,\delta}(\overline{D})$, consisting of all extended mappings, is equicontinuous in $\overline{D}_P$.*