On regular solutions of the Dirichlet problem for the Beltrami equations

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Recall that a **Beltrami equation** in a domain $D \subseteq \mathbb{C}$ is an equation of the form $f_{\bar{z}} = \mu(z)\, f_z$, where $\mu : D \to \mathbb{C}$ is a measurable function with $|\mu(z)| < 1$ a.e., $f_{\bar{z}} = (f_x + if_y)/2$, $f_z = (f_x - if_y)/2$, $z = x + iy$, and f_x and f_y are partial derivatives of f in x and y, correspondingly. The Beltrami equation is said to be **degenerate** if $\operatorname{ess\,sup} K_\mu(z) = \infty$, where $K_\mu(z) = (1 + |\mu(z)|)/(1 - |\mu(z)|)$. Note that the Beltrami equation is a complex form of one of the main equations of the mathematical physics in anisotropic and inhomogeneous media.

A **regular solution** of the Beltrami equation is a continuous, discrete and open mapping $f : D \to \mathbb{C}$ of the Sobolev class $W^{1,1}_{\rm loc}$ with its Jacobian $J_f(z) = |f_z|^2 - |f_{\bar{z}}|^2 \neq 0$ a.e. satisfying the equation a.e. Recall that a mapping $f : D \to \mathbb{C}$ is called **discrete** if the preimage $f^{-1}(y)$ consists of isolated points for every $y \in \mathbb{C}$, and **open** if f maps every open set $U \subseteq D$ onto an open set in $\mathbb{C}$.

We show that the Dirichlet problem with continuous boundary data in Jordan domains has regular solutions for a wide circle of the degenerate Beltrami equations, see many criteria in [1]–[4], for instance:

Theorem 1. *Let D be a Jordan domain in $\mathbb{C}$, $\varphi : \partial D \to \mathbb{R}$ be a continuous function, $\varphi(\zeta) \not\equiv$ const, and $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. satisfying the integral condition*

$$\int\limits_D \Phi\left(K_\mu(z)\right)\, dm(z) < \infty \tag{1}$$

for a convex non-decreasing function $\Phi : [0,\infty] \to [0,\infty]$ such that, for some $\delta > 0$,

$$\int\limits_\delta^\infty \log \Phi(t)\, \frac{dt}{t^2} \; = \; +\infty\ . \tag{2}$$

Then the Beltrami equation has a regular solution with $\lim\limits_{z\to\zeta} \operatorname{Re} f(z) = \varphi(\zeta)$ for all the points $\zeta \in \partial D$.

Corollary 2. *In particular, the conclusion of Theorem 1 holds if, for some* $\alpha > 0$,

$$\int_D e^{\alpha K_\mu(z)}\, dm(z) < \infty\,. \tag{3}$$

References

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