Extremal decomposition of the complex plane

Alexandr Bakhtin, Iryna Denega
(Institute of mathematics of the National Academy of Sciences of Ukraine, Department of complex analysis and potential theory, 3 Tereschenkivska St, Kyiv, Ukraine, 01024)
E-mail: `abahtin@imath.kiev.ua, iradenega@gmail.com`

Let $\mathbb{N}$, $\mathbb{R}$ be the sets of natural and real numbers, respectively, let $\mathbb{C}$ be the complex plane, and let $\overline{\mathbb{C}} = \mathbb{C}\bigcup\{\infty\}$ be its one-point compactification, $\mathbb{R}^+ = (0, \infty)$. Let $r(B, a)$ be the inner radius of the domain $B \subset \overline{\mathbb{C}}$ relative to a point $a \in B$. The inner radius of the domain B is connected with Green's generalized function $g_B(z, a)$ of the domain B by the relations

$$g_B(z, a) = -\ln|z - a| + \ln r(B, a) + o(1), \quad z \to a,$$
$$g_B(z, \infty) = \ln|z| + \ln r(B, \infty) + o(1), \quad z \to \infty.$$

Definition 1. Let $n \in \mathbb{N}$, $n \geqslant 2$. The system of points $A_n := \left\{a_k \in \mathbb{C} :\ k = \overline{1, n}\right\}$ is called n-ray, if $|a_k| \in \mathbb{R}^+$ for $k = \overline{1, n}$ and $0 = \arg a_1 < \arg a_2 < \ldots < \arg a_n < 2\pi$.

Denote $\alpha_k := \dfrac{1}{\pi} \arg \dfrac{a_{k+1}}{a_k}$, $\alpha_{n+1} := \alpha_1$, $k = \overline{1, n}$, $\sum\limits_{k=1}^{n} \alpha_k = 2$.

Problem 2. (V.N. Dubinin [1, 2]) For all values of the parameter $\gamma \in (0, n]$ to show that the maximum of the functional

$$I_n(\gamma) = r^{\gamma}\left(B_0, 0\right) \prod_{k=1}^{n} r\left(B_k, a_k\right),$$

where B_0, B_1, B_2,..., B_n, $n \geqslant 2$, are pairwise disjoint domains in $\overline{\mathbb{C}}$, $a_0 = 0$, $|a_k| = 1$, $k = \overline{1, n}$, is attained for the configuration of domains B_k and points a_k which possesses the n-fold symmetry.

In work [1], the above-formulated problem was solved for the value of the parameter $\gamma = 1$ and all values of the natural parameter $n \geqslant 2$. Namely, it was shown that the following inequality holds

$$r(B_0, 0) \prod_{k=1}^{n} r(B_k, a_k) \leqslant r\left(D_0, 0\right) \prod_{k=1}^{n} r\left(D_k, d_k\right),$$

where d_k, D_k, $k = \overline{0, n}$, are the poles and circular domains of the quadratic differential

$$Q(w)dw^2 = -\frac{(n^2 - 1)w^n + 1}{w^2(w^n - 1)^2}\, dw^2.$$

In work [3], L.V. Kovalev got its solution for definite sufficiently strict limitations on the geometry of arrangement of the systems of points on a unit circle, namely, for systems of points for which the following inequalities hold

$$0 < \alpha_k \leqslant 2/\sqrt{\gamma}, \quad k = \overline{1, n}, \quad n \geqslant 5.$$

In work [4], it was shown that the result by L. V. Kovalev is true for $n=4$. The solution of this problem for $\gamma \in (0,1]$ was given in work [5]. Some partial cases of this problem were studied, for example, in [6–10].

For the further analysis, we calculate the quantity

$$I_n^0(\gamma) = r^{\gamma}\left(D_0, 0\right) \prod_{k=1}^{n} r\left(D_k, d_k\right),$$

where d_k, D_k, $k=\overline{0,n}$, $d_0=0$, are, respectively, the poles and circular domains of the quadratic differential

$$Q(w)dw^2 = -\frac{(n^2-\gamma)w^n+\gamma}{w^2(w^n-1)^2}\, dw^2.$$

As was shown in [1, 2, 3, 6], the quantity $I_n^0(\gamma)$ takes the form

$$I_n^0(\gamma) = \left(\frac{4}{n}\right)^n \frac{\left(\frac{4\gamma}{n^2}\right)^{\frac{\gamma}{n}}}{\left(1-\frac{\gamma}{n^2}\right)^{n+\frac{\gamma}{n}}} \left(\frac{1-\frac{\sqrt{\gamma}}{n}}{1+\frac{\sqrt{\gamma}}{n}}\right)^{2\sqrt{\gamma}}.$$

Theorem 3. [9] *Let $\gamma \in (1, 2]$. Then, for any different points a_1 and a_2 of a unit circle and any mutually disjoint domains B_0, B_1, B_2, $a_1 \in B_1 \subset \overline{\mathbb{C}}$, $a_2 \in B_2 \subset \overline{\mathbb{C}}$, $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, the inequality*

$$r^{\gamma}\left(B_0, 0\right) r\left(B_1, a_1\right) r\left(B_2, a_2\right) \leqslant I_2^0(\gamma) \left(\frac{1}{2}\left|a_1 - a_2\right|\right)^{2-\gamma}.$$

is true. The sign of equality in this inequality is attained, when the points a_0, a_1, a_2 and the domains B_0, B_1, B_2 are, respectively, the poles and circular domains of the quadratic differential

$$Q(w)dw^2 = -\frac{(4-\gamma)w^2+\gamma}{w^2(w^2-1)^2} dw^2.$$

Remark 4. Theorem 3 yields the complete solution of the above-posed problem for two free poles located on the unit circle.

Theorem 5. [9] *Let $n \in \mathbb{N}$, $n \geqslant 3$, $\gamma \in (1, n]$. Then, for any system of different points $A_n = \{a_k\}_{k=1}^n \in \mathbb{C}/\{0\}$ of a unit circle and for any collection of mutually disjoint domains B_0, B_k, $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1,n}$, the inequality*

$$r^{\gamma}(B_0, 0) \prod_{k=1}^{n} r(B_k, a_k) \leqslant \left(\sin\frac{\pi}{n}\right)^{n-\gamma} \left(I_2^0\left(\frac{2\gamma}{n}\right)\right)^{\frac{n}{2}}$$

holds.

References

[1] Dubinin V.N. Symmetrization method in geometric function theory of complex variables. *Russian Math. Surveys*, 1 : 1–79, 1994.

[2] Dubinin V.N. *Condenser capacities and symmetrization in geometric function theory.* Birkhäuser/Springer, Basel, 2014.

[3] Kovalev L.V. On the problem of extremal decomposition with free poles on a circle. *Dal'nevostochnyi Mat. Sb.*, (2) : 96–98, 1996. (in Russian)

[4] Bakhtin A.K., Denega I.V. Addendum to a theorem on extremal decomposition of the complex plane. *Bulletin de la société des sciences et des lettres de Łódź, Recherches sur les déformations*, 62(2) : 83–92, 2012.

[5] Kuz'mina G.V. The Method of Extremal Metric in Extremal Decomposition Problems with Free Parameters. *J. Math. Sci.*, 129(3) : 3843–3851, 2005.

[6] Bakhtin A.K., Bakhtina G.P., Zelinskii Yu.B. *Topological-algebraic structures and geometric methods in complex analysis.* Zb. prats of the Inst. of Math. of NASU, 2008. (in Russian).

[7] Bakhtin A.K. Separating transformation and extremal problems on nonoverlapping simply connected domains. *J. Math. Sci.*, 234(1) : 1–13, 2018.

[8] Bakhtin A.K. Extremal decomposition of the complex plane with restrictions for free poles. *J. Math. Sci.*, 231(1) : 1–15, 2018.

[9] Bakhtin A.K., Denega I.V. Extremal decomposition of the complex plane with free poles. *J. Math. Sci.*, 246(1) : 1–17, 2020.

[10] Tamrazov P.M. Extremal conformal mappings and poles of quadratic differentials. *Mathematics of the USSR-Izvestiya*, 2(5) : 987–996, 1968.