Extended symmetry analysis of dispersionless Nyzhnyk equation

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In [3], Leonid Nyzhnyk[1] constructed one of the first (1+2)-dimensional integrable systems of differential equations,

$$w_t = k_1 w_{xxx} + k_2 w_{yyy} + 3(v^1 w)_x + 3(v^2 w)_y, \quad v^1_y = k_1 w_x, \quad v^2_x = k_2 w_y, \tag{1}$$

where $(k_1, k_2) \neq (0,0)$. Later, it was called the Nyzhnyk system. In fact, this is a family of systems parameterized by two constants, k_1 and k_2. distinguishing nonequivalent values of the parameter tuple (k_1, k_2) in the system (1), as well as introducing potentials, taking dispersionless limits, interpreting independent and/or dependent variables as real or complex or, in the complex case, as related by the complex conjugation and using differential substitutions, one can obtain various integrable models.

We carried out extended symmetry analysis of the (real symmetric potential) Nyzhnyk equation

$$u_{txy} = (u_{xx}u_{xy})_x + (u_{xy}u_{yy})_y, \tag{2}$$

which is also called as the dispersionless Nyzhnyk–Novikov–Veselov equation or even the dispersionless Novikov–Veselov equation. This equation is the dispersionless counterpart of the real symmetric potential Nyzhnyk equation. In the presented full name of the equation, the attribute "real" means that both the independent and dependent variables in the equation are real. The choice of the basic field for the variables is important since the point and contact symmetry groups of the equation depend on it.

In the literature, there had been a number of attempts to study the equation (2) within the framework of symmetry analysis of differential equations. However, they are usually unsuccessful since the obtained results are not complete or reliable. Therefore, it had been important for one to perform the symmetry analysis of the dispersionless Nyzhnyk equation correctly and optimally, applying modern methods of symmetry analysis and using suitable terminology.

Simultaneously with the equation (2), we considered its nonlinear Lax representation

$$v_t = \frac{1}{3}\left(v_x^3 - \frac{u_{xy}^3}{v_x^3}\right) + u_{xx}v_x - \frac{u_{xy}u_{yy}}{v_x}, \quad v_y = -\frac{u_{xy}}{v_x}, \tag{3}$$

and the dispersionless counterpart

$$p_t = (h^1 p)_x + (h^2 p)_y, \quad h^1_y = p_x, \quad h^2_x = p_y \tag{4}$$

[1] A more common transliteration of this surname in the literature is "Nizhnik", as used in [1, 4]. However, the variant "Nyzhnyk" used in [5] is more accurate, given that it corresponds to Ukrainian transliteration rules. Hence, we use it hereafter.

of the symmetric Nyzhnyk system.

In [1], we studied symmetry properties of the equation (2) and the systems (3) and (4). In particular, we found their maximal Lie invariance algebras $\mathfrak{g}$, $\mathfrak{g}_{\rm L}$ and $\mathfrak{g}_{\rm dN}$ and the maximal contact-symmetry algebra $\mathfrak{g}_{\rm c}$ of the equation (2).

The structure of these algebras was analyzed, which includes constructing the sets of their megaideals that suffice for further consideration. The basic among their megaideals are their radicals $\mathfrak{r}$, $\mathfrak{r}_{\rm L}$ and $\mathfrak{r}_{\rm dN}$. One of the required megaideals of $\mathfrak{g}_{\rm L}$ cannot be found by standard methods. Therefore, we developed a new method of constructing megaideals, which was used in this case. In addition, the algebra $\mathfrak{g}_{\rm c}$ is the first prolongation of the algebra $\mathfrak{g}$, and the algebras $\mathfrak{g}_{\rm L}$ and $\mathfrak{g}_{\rm dN}$ are prolongations of $\mathfrak{g}$ to the pseudopotential v and to the tuple of potentials (p,q), respectively.

Applying an original megaideal-based version of the algebraic method, we computed the point-symmetry pseudogroups G, $G_{\rm L}$ and $G_{\rm dN}$ of the equation (2) and the systems (3) and (4), respectively, as well as the contact-symmetry pseudogroup $G_{\rm c}$ of the equation (2). It turned out that the necessary algebraic condition, which is the base of the method, completely defines the pseudogroup G, and therefore there is no need to use the direct method for completing the computation. This is the first example of this kind in the literature. In addition, we proved that the pseudogroup G contains exactly three independent discrete elements, and the pseudogroup $G_{\rm c}$ is the first prolongation of G. The computation of the pseudogroup $G_{\rm c}$ is the first example of applying the megaideal-based version of the algebraic method to finding the contact-symmetry pseudogroup of a system of differential equations. Unlike continuous point symmetries, not all discrete point symmetries of the equation (2) can be prolonged to those of the system (3). The algebraic parts of the computations of the pseudogroups $G_{\rm L}$ and $G_{\rm dN}$ are quite similar to their counterpart for the pseudogroup G. At the same time, since a number of restrictions for the components of point symmetry transformations cannot be derived within the framework of the algebraic method, the role of the direct method becomes more significant here (especially for the pseudogroup $G_{\rm dN}$) than in the course of constructing the pseudogroup G.

In connection with the indicated peculiarity of applying the algebraic method to the equation (2) and for a deeper understanding of the background of this method, we checked whether the finite-dimensional subalgebras $\mathfrak{s}_1$ and $\mathfrak{s}_2$ of the algebra $\mathfrak{g}$, which naturally arise in the course of the above computation of G, define the diffeomorphisms stabilizing this algebra or its first prolongation.

Definition. We call a proper (pseudo)subalgebra $\mathfrak{s}$ of a Lie (pseudo)algebra $\mathfrak{a}$ of (partial) vector fields a *(pseudo)subalgebra defining the (local) diffeomorphisms that stabilize* $\mathfrak{a}$ if the conditions $\Phi_*\mathfrak{a}\subseteq\mathfrak{a}$ and $\Phi_*\mathfrak{s}\subseteq\mathfrak{a}$ for (local) diffeomorphisms Φ in the underlying manifold are equivalent.

The above study gave unexpected results. In particular, the subalgebra $\mathfrak{s}_2$ defines the local diffeomorphisms that stabilize $\mathfrak{g}$, whereas the subalgebra $\mathfrak{s}_1$ and even the subalgebra $\bar{\mathfrak{s}}_1$, which is the natural extension of the subalgebra $\mathfrak{s}_1$ by a vector field from $\mathfrak{g}$, do not have this property. Similarly, the first prolongation of the extension of the subalgebra $\mathfrak{s}_2$ by three linearly independent vector fields from $\mathfrak{g}$, which is a subalgebra of the algebra $\mathfrak{g}_{\rm c}=\mathfrak{g}_{(1)}$, defines the local diffeomorphisms of the corresponding first-order jet space that stabilize $\mathfrak{g}_{\rm c}$. Moreover, this study contains the alternative construction of the pseudogroups G and $G_{\rm c}$ based on the primitive version of the algebraic method. The corresponding computations are much more complicated than those in the course of using the megaideal-based version of the algebraic method, which justifies the application of the latter version in general.

We described all the third-order partial differential equations in three independent variables that are invariant with respect to the algebra $\mathfrak{g}$. We also find a set of geometric properties of the equation (2) that singles out it from the entire class of third-order partial differential equations with three independent variables. In addition to the invariance with respect to the algebra $\mathfrak{g}$, it includes the presence of the conservation-law characteristics 1, u_{xx} and u_{yy}. This combines an inverse group classification problem with an inverse problem on conservation laws.

In [4], the Lie reductions of the equation (2) were exhaustively studied and the wide families of its invariant solutions were constructed. Therein, we also presented for the first time a precise and formalized description of the complete optimized Lie reduction procedure in the case of a system of partial differential equations with three independent variables, which is relevant to the equation (2).

More specifically, using the results of [1], we classified one- and two-dimensional subalgebras of the algebra $\mathfrak g$ and one-dimensional subalgebras of the algebra $\mathfrak g_{\rm L}$ up to the G- and $G_{\rm L}$-equivalences, respectively. Instead of the standard approach, which is based on finding and using the inner automorphisms of Lie algebras, we considered the action of the pseudogroup G on the algebra $\mathfrak g$, which was found by pushing forward vector fields from $\mathfrak g$ by elements of the pseudogroup G. This method is more convenient for computing in the case of infinite-dimensional Lie algebras. In addition, in the course of classifying subalgebras, it allows one to take into account not only continuous, but also discrete point symmetry transformations of the equation (2), which makes it possible to reduce the corresponding optimal lists of subalgebras. The constructed lists of subalgebras created a basis for efficiently and exhaustively carrying out Lie reductions of the equation (2) to partial differential equations with two independent variables and to ordinary differential equations.

When performing the Lie reduction procedure for the equation (2), we observed for the first time several interesting phenomena. In particular, the reduced equations inherit not all the parameters of the corresponding families of inequivalent subalgebras. The utmost for this phenomenon is the case when all inequivalent subalgebras from a family even parameterized by arbitrary functions correspond, under an appropriate choice of ansatzes, to the same reduced equation. Another display of this phenomenon is the possibility of mapping a class of reduced equations to its subclass, which has a less number of parameters. Some equivalent two-dimensional subalgebras of the algebra $\mathfrak g$ with a nonzero one-dimensional intersection induce inequivalent one-dimensional subalgebras of the maximal Lie invariance algebra of a reduced partial differential equation that is obtained by the Lie reduction with respect to the intersection. The algebra $\mathfrak g$ is embedded in the algebra $\mathfrak g_{\rm L}$ via prolonging the vector fields from $\mathfrak g$ to the pseudopotential v, and thus any Lie reduction of the equation (2) has a counterpart among Lie reductions of the system (3) but such a counterpart is in general not unique even up to the $G_{\rm L}$-equivalence. Moreover, in contrast to Lie symmetries, simple and obvious discrete point symmetries of the equation (2), even under the optimal choice of ansatzes, can induce complicated and nontrivial discrete point symmetries of the corresponding reduced equations.

We computed for the first time the point symmetry groups of reduced equations, including their discrete point symmetries, and it was checked in all the cases whether these symmetries are hidden or induced. Since most of the obtained reduced equations for the equation (2) are quite cumbersome, various versions of the algebraic method are much more efficient in the course of the above computation than the direct method. In addition, some of these reduced equations are not of maximal rank. Therefore, the mentioned analysis of reduced equations is, in particular, the first explicit and systematic study of Lie and general point symmetries of differential equations that are not of maximal rank. It is also deeper than its analogues in most papers in the field of classical group analysis: we applied a wider set of methods and techniques, solved an unusually large proportion of reduced equations, and more systematically studied the hidden symmetries of the original equation. For integrating and finding exact solutions of some reduced ordinary differential equations of the equation (2), we involved the corresponding Lie reductions of its nonlinear lax representation (3). As a result, we constructed wide families of new invariant solutions of the equation (2) in explicit form in terms of elementary, Lambert and hypergeometric functions as well as in parametric or implicit form. In addition, we showed that

Lie reductions of the equation (2) to algebraic equations give no new solutions of this equation as compared to the already constructed ones.

Since any function of the form $u = w(t,x) + \tilde{w}(t,y)$, which corresponds to the additive separation of the variables x and y, is a solution of the equation (2), this separation of variables is trivial for (2). Therefore, to look for non-Lie solutions of the equation (2) that generalize some of its invariant solutions, we used the multiplicative separation of the variables x and y, the ansatz for which has the form $u = \varphi(t,x)\psi(t,y)$ with $\varphi_x \neq 0$ and $\psi_y \neq 0$. The obtained results show that more closed-form solutions of (2) can be constructed using other tools of symmetry analysis of differential equations.

The list of inequivalent one-dimensional subalgebras of the maximal Lie invariance pseudoalgebra $\mathfrak{g}$ of (2) presented in [4] includes, in particular, the family of subalgebras $\mathfrak{s}^{\rho}_{1.3} = \big\langle \partial_x + \rho\partial_y - \frac12\rho_t y^2\partial_u \big\rangle$, where $\rho = \rho(t)$ is an arbitrary smooth function of t satisfying the inequalities $\rho(t) \neq 0$ for all t in its domain and $\rho \not\equiv 1$ on any open interval within that domain. The optimal ansatzes constructed with respect to these subalgebras reduce the equation (2) to partial differential equations in two independent variables that share the same form

$$w_{122} + w_{22}w_{222} = 0, \tag{5}$$

which we call reduced equation 1.3 due to its association with the subalgebra family $\{\mathfrak{s}^{\rho}_{1.3}\}$, marking by “1.3” all the object related to this equation. It is easy to see that the substitution $w_{22} = h$ maps the equation (5) to the inviscid Burgers equation

$$h_1 + hh_2 = 0. \tag{6}$$

The equation (5) is the most interesting and fruitful submodel of (2), in particular, in the sense of its relation to hidden symmetry-like objects of (2). This is why it was comprehensively studied in [5] with the framework of extended symmetry analysis of differential equations. After finding the maximal lie invariance algebra $\mathfrak{g}_{1.3}$ of the equation (5) and constructing a set of essential megaideals of this algebra, we applied an original megaideal-based version of the algebraic method [2] to compute the point-symmetry pseudogroup $G_{1.3}$ of the equation (5). It turned out that this pseudogroup has the same remarkable property as the pseudogroup G has. namely, the algebraic condition that the pushforward Φ_* of the algebra $\mathfrak{a}_{1.3}$ by any element Φ of $G_{1.3}$ preserves this algebra, $\Phi_*\mathfrak{a}_{1.3} = \mathfrak{a}_{1.3}$, completely defines the pseudogroup $G_{1.3}$. Therefore, the direct method was needed only to verify that the pseudogroup $G_{1.3}$ is indeed the entire point-symmetry pseudogroup of (5). After the pseudogroup G [1], this was the second but much simpler example of this kind in the literature, and no more examples are known.

Inspired by finding the above phenomenon, we studied other defining properties of Lie symmetries of the equation (5). In particular, we proved that this equation is Lie-remarkable since it itself is completely defined by 11- and 12-dimensional subalgebras of the algebra $\mathfrak{a}_{1.3}$ in the classes of genuine and general partial differential equations of order not greater than three in two independent variables, respectively, whereas a six-dimensional subalgebra of the former subalgebra suffices to define the local diffeomorphisms that stabilize the algebra $\mathfrak{a}_{1.3}$. Beside this, we studied the induction of Lie and point symmetries of the reduced equation (5) by their counterparts for the original equation (2). Jointly with [4], this gave the first examples of studying the induction of point symmetries, including discrete ones, in the course of a Lie reduction, which is a more complicated problem than that of inducing Lie symmetries.

In [5], we also classified, up to the $G_{1.3}$-equivalence, one-dimensional subalgebras of $\mathfrak{a}_{1.3}$ that are appropriate for Lie reduction of the equation (5). This classification was carried out by means of reducing it to the classification of one-dimensional subalgebras of the algebra $\check{\mathfrak{a}}_{1.3}$ up to the $\check{G}_{1.3}$-equivalence. Here $\check{\mathfrak{a}}_{1.3}$ denotes the algebra of Lie-symmetry vector fields of (6) that are induced by Lie-symmetry vector fields of (5). Analogously, $\check{G}_{1.3}$ denotes the group of point symmetry transformations of (6) that are induced by point symmetry transformations of (5). This created a basis for the exhaustive classification of Lie reductions of the equation (5) and finding large families of Lie invariant solutions of the equations (6) and (5).

In addition, we found all local symmetry-like objects associated with the equation (5), including generalized symmetries, cosymmetries, conservation-law characteristics and conservation laws, and most of them are hidden for the original equation (2). This represents the first comprehensive study of such objects for a submodel of a well-known system of differential equations. Complete descriptions even of particular kinds of such objects in nontrivial cases exist in the literature only for a minor part of these systems themselves, not to mention their submodels. Moreover, a complete description of all the local symmetry-like objects of a model in a single paper is rather exceptional.

Standard techniques like recursion operators and the estimation of the dimension of the space of objects in question up to an arbitrary fixed order do not work for the equation (5). Even the best computer packages for finding local symmetry-like objects such as Jets and GeM for Maple are inefficient at computing such objects for this equation even at low orders, starting from order three. This can be explained by the fact that for local symmetry-like objects of any specific kind, the corresponding space of them for the equation (5) is of complicated structure. In particular, they are parameterized by functions of arbitrary finite number of arguments that are cumbersome differential expressions.

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