On conformal differential invariants of plane 3-webs

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We continue to study differential invariants for plane 3-webs, with respect to infinite dimensional Lie pseudogroups. The case of the pseudogroup of local symplectomorphisms was studied in [1] while in [2] we investigated the orbits of plane 3-webs, with respect to the pseudogroup of local conformal diffeomorphisms. We'll use here notations of the paper [2].

Let $D \subset \mathbb{R}^2$ be an open and connected domain. A 3-web W_3 in the domain is defined by differential 1-forms $W_3 = \langle \omega_1, \omega_2, \omega_3 \rangle$, being in general position, i.e. $\omega_i \wedge \omega_j \neq 0$, for $i \neq j$.

The 3-web itself consist of three families of integral curves for forms $\omega_1, \omega_2, \omega_3$.

We see also, that triples of differential 1-forms $\omega_1, \omega_2, \omega_3$ and $\omega_1', \omega_2', \omega_3'$ determine the same 3-web if and only if $\omega_i = f_i \omega_i'$, $i = 1, 2, 3$, for smooth nowhere-vanishing functions f_i nowhere vanishing in D.

We have actions of the following groups on the 3-webs, represented as $W_3 = \langle \omega_1, \omega_2, \omega_3 \rangle$, for normalized triples $\langle \omega_1, \omega_2, \omega_3 \rangle$.

1. The group S_3 of permutations of three letters acts in the natural way:

$$\sigma : W_3 \left\langle \omega_1, \omega_2, \omega_3 \right\rangle \rightarrow W_3 \left\langle \omega_{\sigma(1)}, \omega_{\sigma(2)}, \omega_{\sigma(3)} \right\rangle,$$

   where $\sigma \in S_3$.

2. The group of scaling transformations Sc consist of nowhere vanishing in the domain D functions $f \in C^{\infty}(D)$, that act on webs in the following way:

$$f : W_3 \left\langle \omega_1, \omega_2, \omega_3 \right\rangle \rightarrow W_3 \left\langle f\omega_1, f\omega_2, f\omega_3 \right\rangle.$$

3. The action of conformal pseudogroup itself.

We will find S_3-invariants of 3-webs by using functions λ_i from [2].

We remark that the elementary symmetric functions

$$\begin{aligned} J_1 &= \lambda_1 + \lambda_2 + \lambda_3, \\ J_2 &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3, \\ J_3 &= \lambda_1\lambda_2\lambda_3 \end{aligned}$$

are S_3-invariants.

They do separate S_3-orbits, when the resultant

$$\Delta (J_1, J_2, J_3) = -4J_1^3 J_3 + J_1^2 J_2^2 + 18 J_1 J_2 J_3 - 4J_2^3 - 27 J_3^2$$

of the polynomial,

$$P = z^3 - J_1 z^2 + J_2 z - J_3, \tag{1}$$

does not vanish.

The scaling transformations $S_{h,f}$ act in the following way on invariants J:

$$S_{h,f} : (J_1, J_2, J_3) \rightarrow \left(s^2 J_1, s^4 J_2, s^6 J_3\right),$$

and

$$S_{h,f} : \Delta \to s^{12}\Delta.$$

Therefore, the curve $\Delta^{-1}(0)$ depends on the 3-web but not on the representation $W_3\langle\omega_1, \omega_2, \omega_3\rangle$.

Thus, we shrink the domain D, if it is necessary, and will assume that $\Delta \neq 0$ in D.

Remark, that polynomial (1) has 3 distinct real roots λ_i in the domain, and we'll number them in the domain, as well as forms ω_i, in such a way, that $\lambda_1 > \lambda_2 > \lambda_3$.

Summarizing, we get the following statement.

Proposition 1. *Let 3-web $W_3\langle\omega_1, \omega_2, \omega_3\rangle$ be defined in domain D on the conformal plane $\left(\mathbb{R}^2, g\right)$, where the discriminant Δ does not vanish anywhere. Then there is a unique way to number forms $\langle\omega_1, \omega_2, \omega_3\rangle$.*

As we have seen, S_3 invariants J_1, J_2, J_3 are relative invariants of the scale transformations. To get scale invariants we remark that we have $J_2 > 0$ in the domain. Therefore, functions

$$I_1 = \frac{J_1}{J_2^{1/2}}, I_2 = \frac{J_3}{J_2^{3/2}}$$

are S_3-invariants as well as scale invariants, and they are determined solely by the conformal structure.

Remark also that polynomial (1) will take now the invariant form:

$$\widetilde{P}(y) = y^3 - I_1 y^2 + y - I_2,$$

for new variable $y = z/J_2^{1/2}$.

Summarizing, we get the main result.

Theorem 2. *Let 3-web $W_3\langle\omega_1, \omega_2, \omega_3\rangle$ be defined in a domain D on the conformal plane $\left(\mathbb{R}^2, g\right)$. Then the functions I_1 and I_2 are conformal invariants.*
Moreover, if the discriminant

$$\widetilde{\Delta} = -4I_1^3 I_2 + I_2^2 + 18 I_1 I_2 - 27 I_2^2 - 4,$$

of the polynomial $\widetilde{P}(y)$ is not vanish anywhere in the domain D, then the 1-forms admit a natural numbering $\omega_1, \omega_2, \omega_3$.

References

[1] Konovenko N., On symplectic invariants of planar 3-webs, *Proc. Intern. Geom. Center*, **15(1)** (2022), 66–74.

[2] Konovenko N.G., On conformal differential invariants and conformal equivalence of planar 3-webs, *J. Geom. Phys.*, **214** (2025), 105516.