Algebraic approach to group classification of nonlinear heat and Burgers equations

Sofiia T. HURAKA

University of Alberta, Edmonton, Canada
E-mail: *huraka@ualberta.ca*

Differential equations are an essential branch of modern mathematics since they are applicable in many spheres. One of important aspects of them is the group analysis of differential equations. The core definition of the group analysis of differential equations is the concept of symmetry – transformations which map solutions of equations (or systems of equations) into solutions of the same equation (or system of equations).

In 18th century symmetries were considered as specific mappings, thus, the process of finding them resulted in search for corresponding groups. At that time this task was fully non-linear, thus, extremely complicated. In the 19th century, however, the Norwegian mathematician Sophus Lie proposed a different approach for computing symmetries. Namely, he connected the process of finding symmetries with corresponding tangent bundles on corresponding groups. Thanks to introducing this idea, Sophus Lie transferred the problem of finding symmetries into linear task of solving overdefined systems of differential equations, thus, simplifying the initial task a lot. However, due to absence of effective methods of solving complicated overdefined systems of differential equations, even infinitesimal method can sometimes lead to extremely cumbersome calculations, especially when the initial task is high-dimensional or involves arbitrary parameters. The process of solving can be simplified even more with the use of different algebraic methods, which are currently being developed, in particular, by people, working at the Kyiv scientific school of symmetry analysis.

The main object of this talk are evolution equations

$$u_t = H(t, x, u, u_1, u_2, \dots, u_r), \tag{1}$$

where t and x are independent variables (time and space variables, respectively); $u = u(t, x)$ is a dependent variable; $u_t = \frac{\partial u}{\partial t}$ is a partial derivative with respect to variable t; $u_k = \frac{\partial^k u}{\partial x^k}$ is k-th partial derivative with respect to variable x, $k = 1, \dots, r$, $r \in \mathbb{N}$, $r \geq 2$, with r denoting the order of equation; H is an arbitrary smooth function of variables $t, x, u, u_1, u_2, \dots, u_r$, $H_{u_r} \neq 0$.

The following lemma (see, for example, [8, Theorem 1] and [3, Lemma 4.5, p. 266]) is widely known.

Lemma 1. *For arbitrary evolution equation of the form* (1)*, the t-component of any infinitesimal operator, corresponding to one-parametric Lie group of local transformations of symmetries of this equation, does not depend on x and u.*

To sum up, one can look for the vector fields, which belong to maximal Lie invariance algebra $\mathfrak{g}^H$ of evolution equations from the class (1), in the form

$$Q = \xi^0(t)\partial_t + \xi^1(t, x, u)\partial_x + \eta(t, x, u)\partial_u, \tag{2}$$

where $\xi^0(t)$, $\xi^1(t, x, u)$ and $\eta(t, x, u)$ run through the set of smooth functions of their arguments.

The classical Lie theorem on Lie algebras of vector fields on the real line is widely known: [7, Satz 6, S. 455] (see also [10, Theorem 2.70], [2, Theorem 1] and [9, Theorem 1.1, p. 26]).

Theorem 2. *Nonequivalent realizations of finite-dimensional Lie algebras by vector fields on the t-line are exhausted by the algebras*

$$\{0\}, \quad \langle\partial_t\rangle, \quad \langle\partial_t,\, t\partial_t\rangle, \quad \langle\partial_t,\, t\partial_t,\, t^2\partial_t\rangle.$$

Denote by π the projection from $\mathbb{R}^t \times \mathbb{R}^x$ onto $\mathbb{R}^t$ and let $k := \dim \pi_* \mathfrak{g}^H$. Since the t-component of the vector fields (2) depends only on the variable$t$, then, according to Theorem 2, $k \leqslant 3$.

At first, based on the Theorems, mentioned above, we clarified and simplified the proof of results of Akhatov, Gazizov and Ibragimov [1, Section 4] on the group classification of the class of evolution equations of the form

$$u_t = H(u_{xx}), \tag{3}$$

where $u_t = \frac{\partial u}{\partial t}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, and H is an arbitrary smooth function of u_{xx}. If H is linear, then the equation (3) is known as the linear heat equation, and, therefore, the corresponding class (3) is sometimes called the class of nonlinear heat equations. According to the Lemma 1, one can look for vector fields, which belong to maximal Lie invariance algebra $\mathfrak{g}^H$ of evolution equations from class (4), in the form (2). We summarize the obtained results in the following statement.

Theorem 3 (The result of group classification of the class (3), [1, Section 4] and [5,6])**.** *A complete list of $G^\sim$-inequivalent (maximal) Lie-symmetry extensions in the class (3) is exhausted by the following cases:*

0) *general case* $H = H(u_{11})$, $\quad \mathfrak{g}^0 = \langle Q^1, Q^2, Q^3, Q^4, Q^5 \rangle$,

1) $H(u_{11}) = \exp(u_{11})$, $\quad \mathfrak{g}^{\exp(u_{11})} = \langle Q^1, Q^2, Q^3, Q^4, Q^5, Q^{6a} \rangle$,

2) $H(u_{11}) = \ln(u_{11})$, $\quad \mathfrak{g}^{\ln(u_{11})} = \langle Q^1, Q^2, Q^3, Q^4, Q^5, Q^{6b} \rangle$,

3) $H(u_{11}) = u_{11}^k, \quad k \neq 0, \pm\frac{1}{3}, 1$, $\quad \mathfrak{g}^{u_{11}^k} = \langle Q^1, Q^2, Q^3, Q^4, Q^5, Q^{6c} \rangle$,

4) $H(u_{11}) = u_{11}^{1/3}$, $\quad \mathfrak{g}^{u_{11}^{1/3}} = \langle Q^1, Q^2, Q^3, Q^4, Q^5, Q^{6c}, Q^{7c_1} \rangle$,

5) $H(u_{11}) = u_{11}^{-1/3}$, $\quad \mathfrak{g}^{u_{11}^{-1/3}} = \langle Q^1, Q^2, Q^3, Q^4, Q^5, Q^{6c}, Q^{7c_2} \rangle$.

The next table presents the results of the group classification of the class (3).

Table 1. The result of group classification of the class (3).

$H(u_{xx})$	Lie invariance algebra
$\forall$	$\langle \partial_t,\, \partial_x,\, \partial_u,\, 2t\partial_t + x\partial_x + 2u\partial_u,\, x\partial_u \rangle$
$\exp u_{xx}$	$\langle \partial_t,\, \partial_x,\, \partial_u,\, 2t\partial_t + x\partial_x + 2u\partial_u,\, x\partial_u, 2t\partial_t - x^2\partial_u \rangle$
$\ln u_{xx}$	$\langle \partial_t,\, \partial_x,\, \partial_u,\, 2t\partial_t + x\partial_x + 2u\partial_u,\, x\partial_u, x\partial_x - 2t\partial_u \rangle$
u_{xx}^k, $k \neq 0, \pm\frac{1}{3}, 1$	$\langle \partial_t,\, \partial_x,\, \partial_u,\, 2t\partial_t + x\partial_x + 2u\partial_u,\, x\partial_u, (1-k)t\partial_t + u\partial_u \rangle$
$u_{xx}^{1/3}$	$\langle \partial_t,\, \partial_x,\, \partial_u,\, 2t\partial_t + x\partial_x + 2u\partial_u,\, x\partial_u, 2t\partial_t + 3u\partial_u, u\partial_x \rangle$
$u_{xx}^{-1/3}$	$\langle \partial_t,\, \partial_x,\, \partial_u,\, 2t\partial_t + x\partial_x + 2u\partial_u,\, x\partial_u,$ $4t\partial_t + 3u\partial_u, x^2\partial_x + xu\partial_u \rangle$

After that, we apply our knowledge for group classification of the class of evolution equations of the following form:

$$u_t + uu_x = H(u_{xx}), \tag{4}$$

where $u_t = \frac{\partial u}{\partial t}$, $u_x = \frac{\partial u}{\partial x}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, H is an arbitrary smooth function of u_{xx}. When H is a linear function, then equation (4) is a well-known Burgers equation.

According to the Lemma 1, one can look for vector fields, which belong to maximal Lie invariance algebra $\mathfrak{g}^H$ of evolution equations from class (4), in the form (2).

Theorem 4 (The result of group classification of the class (4), [4] and [5]). *A complete list of $G^{\sim}$-inequivalent (maximal) Lie-symmetry extensions in the class (4) is exhausted by the following cases:*

(0) *general case* $H = H(u_{11})$,
$\mathfrak{g}^0 = \langle \partial_t,\, \partial_x,\, t\partial_x + \partial_u \rangle$,

(1) $H(u_{11}) = \ln(u_{11})$,
$\mathfrak{g}^{\ln(u_{11})} = \langle \partial_t,\, \partial_x,\, t\partial_x + \partial_u,\, t\partial_t + \big(2x - \frac{3}{2}t^2\big)\partial_x + (u - 3t)\partial_u \rangle$,

(2) $H(u_{11}) = u_{11}^p, \quad p \neq 0, \frac{1}{3}, 1$,
$\mathfrak{g}^{u_{11}^p} = \langle \partial_t,\, \partial_x,\, t\partial_x + \partial_u,\, (p+1)t\partial_t + (2-p)x\partial_x + (1-2p)u\partial_u \rangle$,

(3) $H(u_{11}) = u_{11}^{1/3}$,
$$\mathfrak{g}^{u_{11}^{1/3}} = \langle \partial_t,\, \partial_x,\, t\partial_x + \partial_u,\, 4t\partial_t + 5x\partial_x + u\partial_u,\, u\partial_x,\, (2tu - x)\partial_x + u\partial_u,\, (tu - x)(t\partial_x + \partial_u) \rangle.$$

Results of group classifiation (4) are provided in Table 2.

Table 2. The result of group classification of the class (4).

$H(u_{xx})$	Lie invariance algebra
$\forall$	$\langle \partial_t,\, \partial_x,\, t\partial_x + \partial_u \rangle$
$\ln u_{xx}$	$\langle \partial_t, \partial_x,\, t\partial_x + \partial_u,\, t\partial_t + (2x - \frac{3}{2}t^2)\partial_x + (u - 3t)\partial_u \rangle$
u_{xx}	$\langle \partial_t,\, \partial_x,\, t\partial_x + \partial_u,\, 2t\partial_t + x\partial_x - u\partial_u,\, t^2\partial_t + tx\partial_x + (x - tu)\partial_u \rangle$
u_{xx}^p, $p \neq 0, \frac{1}{3}, 1$	$\langle \partial_t,\, \partial_x,\, t\partial_x + \partial_u,\, (p+1)t\partial_t + (2-p)x\partial_x + (1-2p)u\partial_u \rangle$
$u_{xx}^{1/3}$	$\langle \partial_t,\, \partial_x,\, t\partial_x + \partial_u,\, 4t\partial_t + 5x\partial_x + u\partial_u,$ $u\partial_x,\, (2tu - x)\partial_x + u\partial_u,\, (tu - x)(t\partial_x + \partial_u) \rangle$

The third line in the table corresppnds to the well-known Burgers equation $u_t + uu_x = u_{xx}$, which admits five-dimensional maximal Lie invariance algebra.

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