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Т. 6 № 6

Зміст

DYDAK J. **Coverings and fundamental groups: a new approach** 7

GOLASIŃSKI M., de MELO T. **Cyclic and cocyclic maps and generalized Whitehead products** 22

BALCERZAK B., PIERZCHALSKI A. **Derivatives of skew-symmetric and symmetric vector-valued tensors** 35

PAVEŠIĆ P. **Formal aspects of topological complexity** . 56

GONÇALVES D. L., LIBARDI A. K. M., PENTEADO D., VIEIRA J. P. **Fixed points on trivial surface bundles** 67

BLASZCZYK Z. **Free $\mathcal{A}_4$-actions on products of spheres** 86

KUBARSKI J. **Koszul complexes and Chevalley's theorems for Lie algebroids** 97

KISZKIEL J. M. **The Lefschetz Theorem for multivalued maps** 118

BUIJS U., MURILLO A. **Rational homotopy type of and pointed mapping spaces between spheres** 130

KUPERBERG K. **Periodicity generated by adding machines** 140

GOLASIŃSKI M., RUIZ F. G. **Spheres over finite rings and their polynomial maps** 148

BAUVAL A., GONÇALVES D. L., HAYAT C., ZVENGROWSKI P. **The Borsuk-Ulam Theorem for Double Coverings of Seifert Manifolds** 165

RUSZKOWSKI S. **Time scale version of the Ważewski retract method** 190

BIASI C., MONIS T. F. M. **Weak local Nash equilibrium - part II** 209

SHARKO V., GOL'COV D. **Semi-free R^1 action and Bott map** 224

Збірник праць Ін-ту математики НАН України 2013, том 6, N 6, 7–21

Jerzy Dydak

Coverings and fundamental groups: a new approach

Classical fundamental groups behave reasonably well for Poincaré spaces (i.e.. semy-locally simply connected spaces). One has a construction of the universal covering for such spaces. For arbitrary spaces it is a different matter.
We define monodromy groups $\pi(p, b_0)$ for any map $p : E \to B$ with the path lifting property and any $b_0 \in B$. p is called a $\mathcal{P}$-covering, where $\mathcal{P}$ is a class of Peano spaces (i.e., connected and locally path connected spaces), if it has existence and uniqueness of lifts of maps $f : X \to B$ for any $X \in \mathcal{P}$. For any B there is the maximal $\mathcal{P}$-covering $p_{\mathcal{P}} : B_{\mathcal{P}} \to B$ and its monodromy group is called the $\mathcal{P}$-fundamental group of (B, b_0). In case of $\mathcal{P}$ consisting of all disk-hedgehogs we construct a universal covering theory of all spaces in analogy to the classical covering theory of Poincaré spaces.

1. Introduction

The traditional approach of defining the fundamental group first and then constructing universal coverings works well only for the class of Poincaré spaces. For general spaces there were several attempts to define generalized coverings (see [1], [3], and [12]), yet there is no general theory so far that covers all path connected spaces. In this paper we plan to remedy that by changing the order of things: we define the universal covering first and its group of deck transformations is the new fundamental group of the base space.

The basic idea is that a non-trivial loop ought to be detected by a covering (not by extension over the unit disk): a loop is non-trivial if there is a covering such that some lift of the loop is a non-loop.

So it remains to define coverings: the most natural class is the class of maps that have unique disk lifting property. To make the theory work one needs to add the assumption that path components of pre-images of open sets form a basis of the total space.

2. Coverings and deck transformations

Maps are synonymous with continuous functions.

Definition 2.1. Let $\mathcal{P}$ be a class of spaces. A map $p : E \to B$ has $\mathcal{P}$-**lifting property** if for any $e_0 \in E$ and any map $f : (X, x_0) \to (B, p(e_0))$, where $X \in \mathcal{P}$, there is a map $g : X \to E$ such that $p \circ g = f$ and $g(x_0) = e_0$.

p is a $\mathcal{P}$-**covering** (or a $\mathcal{P}$ **covering**) if it has the $\mathcal{P}$-lifting property and all lifts are unique. That means $g = h$ if $g, h : X \to E$, $p \circ g = p \circ h$, and $g(x_0) = h(x_0)$ for some $x_0 \in X \in \mathcal{P}$.

Of special interest are arc-coverings ($\mathcal{P}$ consists of the unit interval I), disk-coverings ($\mathcal{P}$ consists of the unit disk D^2), and hedgehog-coverings (see 3.1 for the definition of hedgehogs).

Definition 2.2. A topological space X is an **lpc-space** if it is locally path-connected. X is a **Peano space** if it is locally path-connected and connected.

Problem 2.3. *Suppose $p : E \to D^2$ is an arc-covering for some Peano space E. Is p a homeomorphism?*

The most fundamental example of a covering is that of the identity function $id : P(X) \to X$ from the **Peanification** $P(X)$ of X to X (see [3]). $P(X)$ is obtained from X by changing its topology to the one whose basis consist of path-components of open sets in X. $id : P(X) \to X$ is a $\mathcal{P}$-covering for the class $\mathcal{P}$ of all Peano spaces.

Proposition 2.4. *If $p : E \to B$ is an arc-covering and E is path-connected, then the fibers of p are T_1 spaces.*

Proof. A space F is T_1 if each point is closed in it. Equivalently, for any two different points $a, b \in F$ there is an open subset of F containing a but not b.

Suppose $e_0, e_1 \in p^{-1}(b_0)$ are two different points such that every neighborhood of e_0 contains e_1. Choose a path α from e_0 to e_1 in E. Consider the loop β obtained from α by changing the value at 1 from e_1 to e_0. Notice β is continuous ($\beta^{-1}(U) = \alpha^{-1}(U)$ for all open subsets U of E) and is a lift of the same path as α, yet ending at a different point, a contradiction. □

2.1. The monodromy group.

Suppose $p : E \to B$ is an arc-covering and $b_0 \in B$. Any loop α at b_0 induces a function from the fiber $F = p^{-1}(b_0)$ to itself that we denote by $x \to \alpha \cdot x$. Namely, we lift α to $\tilde{\alpha}$ starting at x and we put $\alpha \cdot x = \tilde{\alpha}(1)$. Notice the function $x \to \alpha \cdot x$ is a bijection: it inverse is $x \to \alpha^{-1} \cdot x$, where $\alpha^{-1}(t)$ is defined as $\alpha(1 - t)$ (in other words, α^{-1} is the reverse of α). We say that α acts on F. Notice the composition of α acting on F and β acting on F is the action of the concatenation $\alpha * \beta$ on F. The basic idea is to identify any two loops that act on F the same way.

Definition 2.5. Suppose $p : E \to B$ is an arc-covering and $b_0 \in B$. The **monodromy group** $\pi(p, b_0)$ of p at b_0 is the set of equivalence classes of loops in B at b_0: $\alpha \sim \beta$ if and only for any two lifts $\tilde\alpha$ (of α) and $\tilde\beta$ (of β) one has $\tilde\alpha(1) = \tilde\beta(1)$ if $\tilde\alpha(0) = \tilde\beta(0)$. The group operation is induced by concatenation: $[\alpha] \cdot [\beta] := [\alpha * \beta]$.

Remark 2.6. Notice the above equivalence of loops can be easily extended to the concept of equivalence of paths in B starting at b_0. We will use that equivalence throughout the paper. In particular, by $\alpha \cdot x$ we mean $\tilde\alpha(1)$, where $\tilde\alpha$ is the lift of α starting at x.

Notice $[\alpha]$ is the trivial element of $\pi(p, b_0)$ if and only if all its lifts are loops.

Notice that, if p is a disk-covering, then any null-homotopic loop of (B, b_0) represents the trivial element of $\pi(p, b_0)$ and there is a natural homomorphism $\pi_1(B, b_0) \to \pi(p, b_0)$ that is surjective.

It is easy to show that $\pi(p, b_0)$ and $\pi(p, b_1)$ are isomorphic just as in the case of classical fundamental groups of spaces.

2.2. **The deck transformation group.**

Definition 2.7. Given a map $p : E \to B$ its **deck transformation group** $DTG(p)$ is the group of homeomorphisms $h : E \to E$ such that $p \circ h = h$.

Proposition 2.8. *If $p : E \to B$ is an arc-covering and E is path-connected, then the group of deck transformations $DTG(p)$ of p acts freely on E.*

Proof. Suppose $g(e) = e$ for a deck transformation g. For any $x \in E$ pick a path α from e to x. Both α and $g \circ \alpha$ are lifts of $p \circ \alpha$ originating at e. Therefore $x = \alpha(1) = (g \circ \alpha)(1) = g(x)$ and $g \equiv id_E$. □

Definition 2.9. An arc-covering $p : E \to B$ is **regular** if for any loop α in B all its lifts are either all loops or all non-loops. This is the same as saying that $\pi(B, b_0)$ acts freely on the fiber $F = p^{-1}(b_0)$.

Notice that, if B is path-connected, regularity of p depends only on loops at a specific point. If no loop at $b_0 \in B$ has mixed lifts, then no loop at another point $b \in B$ has mixed lifts.

Proposition 2.10. *If $p : E \to B$ is a regular arc-covering and E is path-connected, then for any $e_0 \in E$ there is a natural monomorphism $DTG(p) \to \pi(p, b_0)$, $b_0 = p(e_0)$. The monomorphism is an isomorphism if $DTG(p)$ acts transitively on the fibers of p.*

Proof. For any $h \in DTG(p)$ choose a path α_h in E from e_0 to $h(e_0)$. Since p is a regular arc-covering, the equivalence class $[p \circ \alpha_h]$ does not depend on the choice of α_h. If $g \in DTG(p)$, then $\alpha_g * g(\alpha_h)$ is a path from e_0 to $g(h(e_0))$ and $[p(\alpha_g * g(\alpha_h))] = [p(\alpha_g) * p(\alpha_h)]$, so it is indeed a homomorphism.

If $DTG(p)$ acts transitively on the fibers of p and $[\alpha] \in \pi(p, b_0)$, then lift α to $\tilde\alpha$ and pick a deck transformation h such that $h(\tilde\alpha(0)) = h(\tilde\alpha(1))$. Notice h is mapped to α. □

Problem 2.11. *Characterize continuous group actions G on a Peano space E such that the projection $p : E \to E/G$ is an arc-covering.*

Problem 2.12. *Characterize continuous group actions G on a Peano space E such that the projection $p : E \to E/G$ is a disk-covering.*

3. Hedgehog coverings

Definition 3.1. A **directed wedge** (see [3]) is the wedge $(Z, z_0) = \bigvee_{s\in S}(Z_s, z_s)$ of pointed Peano spaces indexed by a directed set S and equipped with the following topology (all wedges in this paper are considered with that particular topology):

(1) $U \subset Z \setminus \{z_0\}$ is open if and only if $U \cap Z_s$ is open for each $s \in S$,
(2) U is an open neighborhood of z_0 if and only if there is $t \in S$ such that $Z_s \subset U$ for all $s > t$ and $U \cap Z_s$ is open for each $s \in S$.

A **arc-hedgehog** is a directed wedge $(Z, z_0) = \bigvee_{s\in S}(Z_s, z_s)$ such that each (Z_s, z_s) is homeomorphic to $(I, 0)$. The **standard arc-hedgehog** is the arc-hedgehog over the set of natural numbers N.

A **disk-hedgehog** is a directed wedge $(Z, z_0) = \bigvee_{s\in S}(Z_s, z_s)$ such that each Z_s is homeomorphic to the 2-disk D^2.

A typical construction of an arc-hedgehog and its map to a space X is the following:

Proposition 3.2. *Let $x_0 \in X$. Suppose $\{\alpha_V : I_V = [0,1] \to X\}_{V\in S}$ is a family of paths in X indexed by a basis S of open neighborhoods V of x_0 in X. If $\alpha_V(I) \subset V$ and $\alpha_V(0) = x_0$ for all $V \in S$ and S is ordered by inclusion ($U \le V$ means $V \subset U$), then the natural function $\alpha = \bigvee_{V\in S}\alpha_V : \bigvee_{V\in S}(I_V, 0) \to X$ is continuous.*

Proof. $\alpha^{-1}(U)$ is certainly open if $x_0 \notin U$. If $x_0 \in U$, then $I_V \subset \alpha^{-1}(U)$ for all $V \subset U$, so α is indeed continuous. □

Corollary 3.3. *Suppose $f\colon Y \to X$ is a function from an lpc-space Y. f is continuous if $f \circ g$ is continuous for every map $g\colon Z \to Y$ from an arc-hedgehog Z to Y.*

Proof. Assume U is open in X and $x_0 = f(y_0) \in U$. Suppose for each path-connected neighborhood V of y_0 in Y there is a path $\alpha_V\colon (I, 0) \to (V, y_0)$ such that $\alpha_V(1) \notin f^{-1}(U)$. Notice the wedge $\alpha = \bigvee_{V\in S}\alpha_V$ is a map from an arc-hedgehog to Y by 3.2 (here S is the family of all path-connected neighborhoods of y_0 in Y). Hence $h = f \circ g$ is continuous and there is $V \in S$ so that $I_V \subset h^{-1}(U)$. That means $f(\alpha_V(I)) \subset U$, a contradiction. □

Remark 3.4. If X is first countable (it has a countable basis at each point) in 3.2 or Y is first countable in 3.3, then it is sufficient to use the standard arc-hedgehog only.

Theorem 3.5. *If $p : E \to B$ is an arc-covering, then the following conditions are equivalent:*

a. *p is an arc-hedgehog covering,*
b. *given an open subset U of E containing e_0, there is a neighborhood V of b_0 in B such that the path component of $p^{-1}(V)$ containing e_0 is a subset of U.*

Proof. a) $\Longrightarrow$ b). Suppose, for every neighborhood V of b_0 in B, there is a path α_V in $p^{-1}(V)$ joining e_0 with a point in $E \setminus U$. The function $\alpha = \bigvee_{V\in S}\alpha_V$:

$H = \bigvee_{V \in S} I_V \to E$ is continuous as $p \circ \alpha$ is continuous and α is the only possible lift of $p \circ \alpha$ at e_0. However, the point-inverse of U under α contains e_0 but none of I_V is contained in it, a contradiction.

b) $\Longrightarrow$ a). Suppose $\alpha = \bigvee_{s \in S} \alpha_s : \bigvee_{s \in S} I_s \to B$ is a map of an arc-hedgehog with the base-point mapped to $b_0 = p(e_0)$. The only possible lift β of α must be obtained by lifting each α_s separately. The only issue is the continuity of β at the base-point. Given a neighborhood U of e_0 in E, pick a neighborhood V of b_0 in B with the property that the path component P of $p^{-1}(V)$ containing e_0 is a subset of U. Pick an open subset W of the base-point of H satisfying $W \subset \alpha^{-1}(V)$ so that W is path-connected. Notice $\beta(W) \subset P \subset U$, which means β is continuous at the base-point of H. □

Corollary 3.6. *If B is first countable and $p : E \to B$ is an arc-covering with E being a Peano space, then p is an arc-hedghehog covering.*

Proof. Suppose $b_0 \in B$ and $\{U_n\}$ is a decreasing basis of neighborhoods of b_0 in B. Given $e \in F = p^{-1}(b_0)$ and a neighborhood V of e in E, assume that for every $n \geq 1$ there is a path α_n in $p^{-1}(U_n)$ joining e to a point $e_n \in E \setminus V$. Consider the infinite concatenation $p(\alpha_1) * p(\alpha_1^{-1}) * p(\alpha_2) * p(\alpha_2^{-1}) * \ldots$ which we assume ends at b_0. The lift γ of β starting at e cannot be a loop as $\gamma^{-1}(V)$ does not contain any e_n. So it ends at a different point of F. Pick a neighborhood W of $\gamma(1)$ not containing e (see 2.4). $\gamma^{-1}(W)$ is a neighborhood of 1 in $[0, 1]$. Therefore infinitely many paths α_n lie in W, a contradiction. □

Corollary 3.7. *If $p : E \to B$ is an arc-hedgehog covering and E is a Peano space, then the fibers of p are regular (T_3-spaces) 0-dimensional spaces.*

Proof. By 2.4, fibers of p are T_1-spaces, so, given $x \notin A$ in a fiber F (and A being closed in F), there is an open neighborhood V of $p(x) = p(y)$ such that the path component W of $p^{-1}(V)$ containing x does not intersect A. The restriction $W \cap F$ of W to F is an open-closed subset of F containing x and missing A. □

Corollary 3.8. *Arc-hedgehog coverings $p : E \to B$ are open if both E and B are locally path-connected.*

Proof. Suppose U is open in E and $e_0 \in U$. Put $b_0 = p(e_0)$ and $F = p^{-1}(b_0)$. By 3.5 there is a path-connected neighborhood V of b_0 such that the path-component of e_0 in $p^{-1}(V)$ is a subset of U. Therefore $V \subset p(U)$ (connect e_0 with a path to any point in V and then lift the path - it must be contained in U). □

Here is an important supplement to 2.10:

Theorem 3.9. *Suppose $p : E \to B$ is an arc-hedgehog covering. If E is a Peano space, then p is regular if and only if the deck transformation group $DTG(p)$ acts transitively on the fibers of p.*

Proof. If $DTG(p)$ acts transitively on the fibers of p, then for any two lifts α and β of the same loop in B there is a deck transformation h such that $h \circ \alpha = \beta$. Hence they are either both loops or both non-loops.

Suppose p is regular and $e_1, e_2 \in E$ with $p(e_1) = e_2$. Given $x \in E$ choose a path α_x in E from e_1 to x and let β_x be the path from e_2 to $h(x)$ with the property $p \circ \alpha_x = p \circ \beta_x$. Notice $h(x)$ does not depend on the choice of α_x as p is regular.

The reason h is continuous is that $h \circ f$ is continuous for any map f from an arc-hedgehog to E. Since analogous construction creates the inverse of h, it is a homeomorphism. □

Proposition 3.10. *Suppose $p : E \to B$ is an arc-hedgehog covering of Peano spaces. If B is metrizable, then E is metrizable.*

Proof. Denote r-balls in B centered at b by $B(b,r)$. Define $d(x,y)$ as the infimum of $r > 0$ such that there is a path α from x to y in E with $p(\alpha([0,1])) \subset B(p(x),r) \cap B(p(y),r)$. Clearly, d is symmetric. Also, $d(x,y) = 0$ implies $x = y$. Indeed, $p(x) = p(y)$ and $x \neq y$ would imply existence of a neighborhood U of $p(x)$ in B such that no path in U can be lifted to a path from x to y (see 3.5).

The proof of the Triangle Inequality is left to the reader.

Given $x \in U$, U open in E, find an $r > 0$ such that the path component of $p^{-1}(B(p(x),r))$ containing x is contained in U (see 3.5). Therefore the r-ball of metric d centered in x is contained in U.

Consider the r-ball $B_d(x,r)$ in d centered at $x \in E$. Look at the path-component U of $p^{-1}(B(p(x),r/2))$ containing x. It must be contained in $B_d(x,r)$ which completes the proof. □

Proposition 3.11. *If $p : E \to B$ an arc-hedgehog covering, E is Peano, and B has a countable basis at b_0, then $F = p^{-1}(b_0)$ is a Baire space.*

Proof. Let $\{U_n\}$ be a basis of open sets at b_0 that forms a decreasing sequence. We plan to show that, given a decreasing sequence $\{V_n\}$ of path-components V_n of $p^{-1}(U_n)$, the intersection $F \cap \bigcap\limits_{n=1}^{\infty} V_n$ is not empty. By induction, pick points $e_n \in V_n$ and paths α_n in V_n joining e_n with e_{n+1}. The infinite concatenation $p(\alpha_1) * p(\alpha_2) * \dots$ (its end-point is declared to be b_0) is a path α in U_1. Lift α starting at e_1 and notice the end-point of the lift belongs to $F \cap \bigcap\limits_{n=1}^{\infty} V_n$. □

Remark 3.12. Combining the proofs of 3.10 and 3.11 one can show E is completely metrizable if B is completely metrizable and both E and B are Peano spaces.

Definition 3.13. Suppose $p : E \to B$ is an arc-hedgehog covering of Peano spaces. p is **trivial** at b_0 if there is a connected neighborhood U of b_0 in B such that p maps each component of $p^{-1}(U)$ homeomorphically onto U.

Theorem 3.14. *Suppose $p : E \to B$ is a regular arc-hedgehog covering of Peano spaces. p is trivial at b_0 if and only if the fiber $F = p^{-1}(b_0)$ contains an isolated point.*

Proof. One direction is obvious, so assume F has an isolated point $e \in F$. Choose a connected neighborhood U of b_0 in B such that the path component V of $p^{-1}(U)$ containing e does not intersect $F \setminus \{e\})$ (see 3.5). Notice p maps V homeomorphically onto U. Indeed, $p(V) = U$ (lift a path from b_0 to any $x \in U$ starting from e to arrive at $y \in V$ such that $p(y) = x$) and $p|V$ has to be injective: if $p(y) = p(z) = b$ for two different points $y, z \in V$, then there is a path β in V from y to z such that $p \circ \beta$ is a loop and picking a path γ from e to y in V results in a loop $p(\gamma) * p(\beta) * p(\gamma^{-1})$ in U that has a lift in V starting at e and ending at a different point contrary to $V \cap F = \{e\}$.

Consider another component W of $p^{-1}(U)$. Using 3.9 one can see there is a deck transformation h such that $h(V) = W$. Therefore $p|W : W \to U$ is a homeomorphism as well. □

Proposition 3.15. *If $p : E \to B$ is an arc-hedgehog covering, then p is a disk-hedgehog covering if and only if it is a disk covering.*

Proof. It only suffices to consider the case p is a disk-covering (the other implication is obvious). Given a map $f : H \to B$ from a disk-hedgehog to B and given $e \in E$ in the fiber of p over the base-point there is only one candidate for the lift of f. That candidate must be continuous as otherwise we would generate a map from an arc-hedgehog to B that has no lift at e. □

4. The whisker topology

In this section we are generalizing the whisker topology that was introduced in [3] in a special case.

Definition 4.1. Let B be a space and $b_0 \in B$. Suppose $\sim$ is an equivalence relation on the set of loops in B at b_0 which induces a group structure on the set of equivalence classes via $[\alpha] \cdot [\beta] := [\alpha * \beta]$ with the constant loop at b_0 being the neutral element and $[\alpha]^{-1} = [\alpha^{-1}]$ for all loops α, β at b_0.

The above can be summarized as follows:

1. $\alpha \sim \beta$ and $\gamma \sim \omega$ implies $\alpha * \gamma \sim \beta * \omega$ for all loops $\alpha, \beta, \gamma, \omega$ at b_0,
2. $\alpha * \alpha^{-1} \sim const$ and $\alpha \sim \alpha * const$ for all loops α, where α^{-1} is the reversed path of α.

The above equivalence relation can be extended to an equivalence relation on the set of all paths in B originating at b_0: $\alpha \sim \beta$ means $\alpha(1) = \beta(1)$ and $\alpha * \beta^{-1} \sim const$.

By the **whisker topology** on the space $P(B, b_0, \sim)$ of equivalence classes $[\alpha]$ we mean the topology with the basis $N([\alpha], U)$, U an open set in B containing $\alpha(1)$, consisting of all $[\beta]$ such that $\beta \sim \alpha * \gamma$ for some path γ in U

Theorem 4.2. a. *$P(B, b_0, \sim)$ is a Peano space and the end-point projection $p : P(B, b_0, \sim) \to B$ has arc-lifting property.*

b. *p is an arc-hedgehog covering if and only if it is an arc-covering.*

c. *p is a disk-hedgehog covering if and only if it is an arc-covering and $\alpha \sim const$ for every loop α at b_0 that is null-homotopic.*

Proof. a. Notice $\lambda \in N([\alpha], U) \cap N([\beta], V)$ implies $N([\lambda], U \cap V) \subset N([\alpha], U) \cap N([\beta], V)$, so it is indeed a topology.

Given α at any point of B let α_t be the path equal to α on the interval $[0, t]$ and then being a constant path. If γ is a path in U originating at $\alpha(1)$, then each $[\alpha * \gamma_t] \in N([\alpha], U)$ and $t \to [\alpha * \gamma_t]$ is continuous (indeed, the inverse of $N([\alpha * \gamma_t], V)$ contains the interval around t that is mapped under γ to V). That means $P(B, b_0, \sim)$ is a Peano space. At the same time it implies p has arc-lifting property.

b. Suppose p is an arc-covering, U is open in B, and α is a path in B starting at b_0 and ending at a point in U. It suffices to show $N([\alpha], U)$ is the path component of $p^{-1}(U)$ containing $[\alpha]$. Suppose $\tilde{\gamma}$ is a path in $p^{-1}(U)$ starting at $[\alpha]$. Put $\gamma = p \circ \tilde{\gamma}$ and notice $t \to [\alpha * \gamma_t]$ is another lift of γ. Thus $\tilde{\gamma}(t) = [\alpha * \gamma_t]$ for all t proving that $\tilde{\gamma}$ is a path in $N([\alpha], U)$. In view of 3.5, p is an arc-hedgehog covering.

c. Assume p is an arc-covering and $\alpha \sim const$ for every loop α at b_0 that is null-homotopic. In view of b) and 3.15 it suffices to show p is a disk-covering.

Suppose $f : D^2 \to B$ and α is a path in B from b_0 to $f(d)$ for some $d \in D^2$. Given $x \in D^2$ let β_x be a path in D^2 from d to x. Define $g(x) \in P(B, b_0, \sim)$ as $g(x) = [\alpha * (f \circ \beta_x)]$ and notice $g(x)$ does not depend on the choice of β_x. Given a map $u : H \to D^2$ from an arc-hedgehog, $g \circ u$ is the lift of $f \circ u$, hence it is continuous. Therefore g is continuous. □

Here is an inner description of arc-hedghehog coverings:

Theorem 4.3. *Suppose E is a Peano space. If $p : E \to B$ is an arc-hedgehog covering and $b_0 \in B$, then p is equivalent to the end-point projection $P(B, b_0, \sim) \to B$, where $P(B, b_0, \sim)$ is equipped with the whisker topology.*

Proof. Pick $e_0 \in E$ with $p(e_0) = b_0$ and declare two paths α and β in B originating at b_0 equivalent if $\alpha \cdot b_0 = \beta \cdot b_0$.

Given a point $x \in E$ choose a path α_x in E from e_0 to x and define $h : E \to P(B, e_0)$ by $h(x) = [p \circ \alpha_x]$.

Since $h^{-1}(N([\alpha_x], U))$ is the path-component of $p^{-1}(U)$ containing x, it is open in E and h is continuous.

If U is an open neighborhood of x in E, choose an open neighborhood V of $p(x)$ with the property that the path component W of x in $p^{-1}(V)$ is contained in U. Notice $N([\alpha_x], V) \subset h(W) \subset h(U)$, so h is open. Since h is bijective, it is a homeomorphism. □

5. Supremums of coverings

Two coverings $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ are said to be **equivalent** if there is a homeomorphism $h : E_1 \to E_2$ satisfying $p_2 \circ h = p_1$. It turns out there is a set of coverings over B such that any disk-hedgehog covering over B is equivalent to one from that set. In that sense we may talk about the set of all disk-hedge coverings over B.

In this section we define a partial order on the set of all disk-hedgehog coverings over a fixed path-connected space B and we show this set has a maximum. That maximum plays the role of the universal covering space.

Definition 5.1. Suppose E_1, E_2 are Peano spaces and $p_1 : E_1 \to B$, $p_2 : E_2 \to B$ are disk-hedgehog coverings. We define the inequality $(p_1, e_1) \geq (p_2, e_2)$ of pointed coverings as follows: $p_1(e_1) = p_2(e_2)$ and there is a continuous function $f : E_1 \to E_2$ satisfying $p_2 \circ f = p_1$ and $f(e_1) = e_2$.

We define the inequality of unpointed coverings $p_1 \geq p_2$ as follows: for every points $e_1 \in E_1$ and $e_2 \in E_2$ such that $p_1(e_1) = p_2(e_2)$ we have $(p_1, e_1) \geq (p_2, e_2)$.

Lemma 5.2. *If $(p_1, e_1) \geq (p_2, e_2)$ and $(p_2, e_2) \geq (p_1, e_1)$, then there is a homeomorphism $h : E_2 \to E_1$ such that $h(e_2) = e_1$ and $p_1 \circ h = p_2$.*

Proof. Choose continuous functions $f : E_1 \to E_2$ and $g : E_2 \to E_1$ such that $p_2 \circ f = p_1$, $p_1 \circ g = p_2$ and $f(e_1) = e_2$, $g(e_2) = e_1$. As $p_1 \circ (g \circ f) = p_1$ and $(g \circ f)(e_1) = e_1$, we get $g \circ f = id_{E_1}$. Similarly, $f \circ g = id_{E_2}$. □

Lemma 5.3. *If p_1 is a regular disk-hedgehog covering and $(p_1, e_1) \geq (p_2, e_2)$, then $p_1 \geq p_2$.*

Proof. Choose a continuous function $f : E_1 \to E_2$ such that $p_2 \circ f = p_1$. Notice f is surjective. Given $x_1 \in E_1$ and $x_2 \in E_2$ satisfying $p_1(x_1) = p_2(x_2)$ choose a deck transformation $h : E_1 \to E_1$ so that $h(x_1) \in f^{-1}(x_2)$ (see 3.9). Put $g = f \circ h$ and notice $p_2 \circ g = p_1$, $g(x_1) = x_2$. □

Corollary 5.4. *$p \geq p$ if and only if p is regular.*

Proof. In view of 5.3 it suffices to show p is regular if $p \geq p$. That follows from 3.9 as any $f : E \to E$ satisfying $p \circ f = p$ must be a homeomorphism. □

Definition 5.5. Suppose $\{p_s : E_s \to B\}_{s\in S}$ is a family of disk-hegehog coverings of Peano spaces over a path-connected B and $e_s \in E_s$ so that $p_s(e_s) = b_0$ for all $s \in S$. (p, e) is the **supremum** of $\{(p_s, e_s)\}_{s\in S}$ if $(p, e) \geq (p_s, e_s)$ for all $s \in S$ and (p, e) is the smallest pointed covering with that property.

Definition 5.6. Suppose $\{p_s : E_s \to B\}_{s\in S}$ is a family of disk-hegehog coverings of Peano spaces over a path-connected B and $e_s \in E_s$ so that $p_s(e_s) = b_0$ for all $s \in S$.

The **Peano fibered product** of $\{(p_s, e_s)\}_{s\in S}$ is the pair (p, e), where $p : E \to B$, $e = \{e_s\}_{s\in S}$, and E is the Peanification of the path-component of e in the subset of $\prod\limits_{s\in S} E_s$ consisting of points $\{x_s\}_{s\in S}$ such that $p_s(x_s) = p_t(x_t)$ for all $s, t \in S$. The projection p is defined by $p(\{x_s\}_{s\in S}) = p_t(x_t)$ for any $t \in S$.

Proposition 5.7. *Peano fibered product of a family of pointed disk-hedgehog coverings is the supremum of that family.*

Proof. If $q : E' \to B$ and $(q, e') \geq (p_s, e_s)$ for all $s \in S$, then there are maps $g_s : E' \to E_s$ so that $q = p_s \circ g_s$ and $g_s(e') = e_s$ for each $s \in S$. The collection $\{g_s\}_{s\in S}$ induces a map $g : E' \to E$ satisfying $g(e') = e$ and $p \circ g = q$. Thus $(q, e') \geq (p, e)$.

Suppose $b_0 = p(\{e_s\}_{s\in S})$, $\{e_s\}_{s\in S} \in E$, and $f\colon (H, 0) \to (B, b_0)$ is a map from a disk-hedgehog. Create lifts $f_s : (H, 0) \to (E_s, e_s)$ of f with respect to p_s. That defines a map $f : H \to E$ by $f(x) = \{f_s(x)\}_{s\in S}$ that is a lift of f with respect to p.

That proves existence of lifts - a proof of uniqueness is obvious. □

Proposition 5.8. *If $p : E \to B$ is a disk-hedgehog covering and $e_0 \in E$, then the Peano fibered product of all $p : (E, e) \to (B, p(e_0))$, e ranging over all points in the fiber F of p containing e_0, is regular.*

Proof. Suppose α is a loop in B at $b_0 = p(e_0)$ such that for some $\{x_e\}_{e\in F}$ in the fiber of q, $\alpha \cdot \{x_e\}_{e\in F} = \{x_e\}_{e\in F}$. That means $\alpha \cdot x_e = x_e$ for all $e \in F$.

Since both $\{x_e\}_{e\in F}$ and $\{e\}_{e\in F}$ can be joined by a path in the Peano fibered product, there is a loop β at b_0 in B such that $\beta\cdot\{e\}_{e\in F} = \{x_e\}_{e\in F}$. Thus $\beta\cdot e = x_e$ and $(\alpha * \beta) \cdot e = \beta \cdot e$ for all $e \in F$. Plugging in $\beta^{-1} \cdot e \in F$ for e in the equation $(\alpha * \beta) \cdot e = \beta \cdot e$ gives $\alpha \cdot e = e$ for all $e \in F$. That implies $\alpha \cdot \{y_e\}_{e\in F} = \{y_e\}_{e\in F}$ for all $\{y_e\}_{e\in F}$ in the fiber of q, i.e. q is regular. □

Notice the Peano fibered product of all $z \to z^n$ is the covering $t \to \exp(2\pi ti)$ of reals over the unit circle.

Corollary 5.9. *Every path-connected space B has a maximal disk-hedgehog covering among those with total space being Peano. It is a regular covering.*

Proof. Pick b_0 and consider the space of paths $P(B, b_0)$ in B starting at b_0. For every disk-hedgehog covering $p : E \to B$, E is an image of a function from

$P(B, b_0)$ obtained by lifting paths (the lifts start at $e_0 \in p^{-1}(b_0)$. That means there is a set $\{p_s : E_s \to B\}_{s\in S}$ of disk-hedgehog coverings with the property that for any disk-hedgehog covering $p : E \to B$ there is $s \in S$ and a homeomorphism $h : E \to E_s$ such that $p = p_s \circ h$. We only consider disk-hedgehog with Peano total space. Take the Peano fibered product of $\{p_s : E_s \to B\}_{s\in S}$. It must be a regular disk-hedge covering but it is easier to use 5.8 and produce the maximal covering that is regular. □

6. Hedgehog fundamental group

Definition 6.1. Given a path-connected space B and $b_0 \in B$ define the **hedgehog fundamental group** $\pi(B, b_0)$ of (B, b_0) as the monodromy group $\pi(p, b_0)$, where $p : E \to B$ is the maximal disk-hedgehog covering over B.

Proposition 6.2. *Any map $f : B_1 \to B_2$ of path-connected spaces induces a natural homomorphism from $\pi(B, b_1)$ to $\pi(B_2, f(b_1))$.*

Proof. Let $f(b_1) = b_2$. Consider the maximum disk-hedgehog covering $p_2 : E_2 \to B_2$ and pick $e_2 \in p_2^{-1}(b_2)$. Take the path-component of (b_1, e_2) in $\{(x, y) \in B_1 \times E_2 | f(x) = p_2(y)\}$, Peanify it to get E and let $q : E \to B_1$ be the projection onto the first coordinate. Notice q is a disk-hedgehog covering. Let $p : E_1 \to E$ be the maximum disk-hedgehog covering over E. Notice $p_1 = q \circ p$ is the maximum disk-hedgehog covering over E_1. If a loop α in B_1 at b_1 has all lifts to E_1 that are loops, then all lifts of α to E must be loops. Given a lift β in E_2 of $f \circ \alpha$, the map $t \to (\alpha(t), \beta(t))$ is a lift of α in E. As it is a loop, β must be a loop as well. Consequently, if two loops in B_1 at b_1 are similar, so are their images in B_2 which is sufficient to conclude there is a natural homomorphism from $\pi(B, b_1)$ to $\pi(B_2, f(b_1))$. □

Proposition 6.3. *If $p : E \to B$ is a regular disk-hedgehog covering and $p(e_0) = b_0$, then one has a natural exact sequence*

$$1 \to \pi(E, e_0) \to \pi(B, b_0) \to \pi(p, b_0) \to 1$$

Proof. Choose a maximal disk-hedgehog covering $p_1 : E_1 \to E$ over E, where E_1 is a Peano space. Notice $p \circ p_1$ is a maximal disk-hedgehog covering over B.

The kernel of $\pi(B, b_0) \to \pi(p, b_0)$ consists exactly of loops whose all lifts to E are loops. In particular, the kernel is contained in the image of $\pi(E, e_0) \to \pi(B, b_0)$. Obviously, the image of $\pi(E, e_0) \to \pi(B, b_0)$ is contained in that kernel.

Any loop in E at e_0 that becomes trivial in $\pi(B, b_0)$ must have all lifts in E_1 as loops. That means $\pi(E, e_0) \to \pi(B, b_0)$ is a monomorphism. □

Theorem 6.4. *Suppose $p : E \to B$ is a disk-hedgehog covering of path connected spaces. Suppose $f : X \to B$ is a map from a Peano space, $x_0 \in X$ and $e_0 \in E$ with $f(x_0) = b_0 = p(e_0)$. f has a lift $g : (X, x_0) \to (E, e_0)$ if and only if the image of $\pi(X, x_0) \to \pi(B, b_0)$ is contained in the image of $\pi(E, e_0) \to \pi(B, b_0)$.*

Proof. Only one implication is of interest, so assume the image of $\pi(X, x_0) \to \pi(B, b_0)$ is contained in the image of $\pi(E, e_0) \to \pi(B, b_0)$.

Given a point $x \in X$ pick a path α_x in X from x_0 to x and define $g(x)$ as $\alpha_x \cdot e_0$. $g(x)$ does not depend on the choice of α_x: choosing a different path β_x leads to a loop γ in E at e_0 such that $[\beta_x * \alpha_x^{-1}] = [p\circ\gamma]$ in $\pi(B, b_0)$. Therefore $\beta_x \sim (p\circ\gamma) * \alpha_x$ and $\beta_x \cdot e_0 = ((p \circ \gamma) * \alpha_x) \cdot e_0 = \alpha_x \cdot e_0 = g(x)$.

Given any map $q : H \to X$ from a disk-hedgehog H to X, the composition $g \circ q : H \to E$ is the only possible lift of $f \circ q$, hence it is continuous. By 3.3, g is continuous. □

Corollary 6.5. *Suppose $p : E \to B$ is a disk-hedgehog covering with E being Peano and $e_0 \in E$. $\pi(E, e_0) = 0$ if and only if $p : E \to B$ is the maximal disk hedgehog-covering over B.*

Proof. If p is maximal, then E does not admit any non-trivial disk-hedgehog covering and $\pi(E, e_0) = 0$. If $\pi(E, e_0) = 0$, then given any other disk-hedgehog covering $q : E_1 \to B$ there is a lift $g : E \to E_1$ of q proving p is maximal. □

7. Comparison to the classical fundamental group

As the natural homomorphism $\pi_1(B, b_0) \to \pi(B, b_0)$ is an epimorphism, there are two natural questions:

Problem 7.1. *Characterize the kernel of $\pi_1(B, b_0) \to \pi(B, b_0)$ for path-connected spaces B.*

Problem 7.2. *Characterize path-connected spaces B such that $\pi_1(B, b_0) \to \pi(B, b_0)$ is an isomorphism.*

Since the identity map $P(B) \to B$ from the Peanification of B to B induces isomorphisms of both the classical fundamental group and the hedgehog fundamental group, we will consider both Problems 7.1 and 7.2 for Peano B spaces only. In particular, we differ with [10] in that regard.

Recall B is **shape injective** if the natural homomorphism $\pi_1(B, b_0) \to \check{\pi}_1(B, b_0)$ from the classical fundamental group to the Čech fundamental group is a monomorphism. Papers [11], [7, Corollary 1.2 and Final Remark], [6], and [9] contain results that various classes of spaces are shape injective. We will generalize the concept of shape injectivity as follows:

Definition 7.3. B is **residually Poincaré** if for every loop α in B that is not null-homotopic there is a map $f : B \to P$ such that P is a Poincaré space and $f \circ \alpha$ is not null-homotopic.

Proposition 7.4. *If B is residually Poincaré, then $\pi_1(B, b_0) \to \pi(B, b_0)$ is an isomorphism.*

Proof. Clearly, it is so if B is a Poincaré space as it has the classical universal cover that is simply connected. Given a non-trivial element $[\alpha] \in \pi_1(B, b_0)$ choose $f(B, b_0) \to (P, p_0)$ such that $f \circ \alpha$ is not null-homotopic. If α represents the neutral element of $\pi(B, b_0)$, then $[f \circ \alpha]$ is neutral in $\pi(P, p_0) = \pi_1(P, p_0)$, a contradiction. □

Theorem 7.5. *Suppose $\mathcal{U}$ is an open cover of a paracompact space B consisting of path-connected sets. If, for each $x \in B$, the inclusion $st(x, \mathcal{U}) \to B$ of the star of $\mathcal{U}$ at x induces the trivial homomorphism of $\pi(st(x, \mathcal{U}), x) \to \pi(B, x)$, then $\pi(B, b_0)$ is isomorphic to the fundamental group of the nerve of $\mathcal{U}$ for all $b_0 \in B$.*

Proof. Pick $V_0 \in \mathcal{U}$ containing b_0. For each $V \in \mathcal{U}$ pick $b_V \in V$ ($b_V = b_0$ if $V = V_0$).

Define a map α from the 1-skeleton of the nerve $N(\mathcal{U})$ to B as follows: each vertex V of the nerve is mapped to b_V and each edge VW is mapped to a path α_{VW} in $V \cup W$ joining b_V and b_W.

Given an edge-path in the nerve from V_0 to V followed by a loop around a triangle that belongs to the nerve, then followed back by the path-edge results in a loop that is mapped to the star $st(b_V, \mathcal{U})$ of b_V in $\mathcal{U}$, hence α induces a homomorphism j from $\pi_1(N(\mathcal{U}), V_0)$ to $\pi(B, b_0)$.

Given a loop λ in B at b_0, we can represent it as the concatenation of paths γ_i, $0 \le i \le n$, such that the carrier of γ_i is contained in $V(i) \in \mathcal{U}$, and $V(0) = V_0 = V(n)$. Pick a path ω_i in $V(i)$ joining $\gamma_i(1)$ and $b_{V(i)}$. Notice each γ_i is equivalent to $\omega_{i-1} * \alpha_{V(i-1)V(i)} * \omega_i^{-1}$, so replacing it by that path results in a loop in the image of j that is equivalent to λ. That proves j is an epimorphism.

To show it is a monomorphism, assume there is an edge-loop in the nerve that is mapped to a loop in B being trivial in $\pi(B, b_0)$. Choose a partition of unity $\phi : B \to N(\mathcal{U})$ sending b_0 to $V_0 \in \mathcal{U}$. The composition of $j : \pi(N(\mathcal{U})) \to \pi(B, b_0)$ and the homomorphism induced by ϕ is the identity.

Indeed, for each $V \in \mathcal{U}$ choose a path β_V in $N(\mathcal{U})$ from $\phi(b_V)$ to V that lies in the open star $st(V)$ of V in $N(\mathcal{U})$. Notice, if $V \cap W \neq \emptyset$, then $\beta_V * VW * \beta_W^{-1} * (\phi(\alpha_{VW}))^{-1}$ lies in the union $st(V) \cup st(W)$ of open stars in $N(\mathcal{U})$. As their intersection is contractible, the union is simply connected and the composition of $j : \pi(N(\mathcal{U})) \to \pi(B, b_0)$ and the homomorphism induced by ϕ is the identity. $\square$

Corollary 7.6. *If B is a paracompact Peano space and $\pi(B, b)$ is discrete for all $b \in B$, then for every sufficiently small open cover $\mathcal{U}$ of B, $\pi(B, b)$ is isomorphic to the fundamental group of the nerve of $\mathcal{U}$ for all $b \in B$.*

Proof. By 3.14 every point $b \in B$ has a path-connected neighborhood U_b such that the maximal disk-hedgehog covering $p : E \to B$ has a section over U_b. That implies $\pi(U_b, b) \to \pi(B, b)$ is trivial. Choose a star-refinement $\mathcal{V}$ of $\{U_b\}_{b \in B}$ and apply 7.5 to any refinement $\mathcal{U}$ of $\mathcal{V}$. $\square$

Let us show that the analog of the famous result of Shelah [19] (see also [18]) stating that the fundamental group of a Peano continuum is finitely generated if it is countable not only holds for the hedgehog fundamental groups but it also has a much simpler proof.

Corollary 7.7. *Suppose B is a Peano continuum. If $\pi(B, b_0)$ is countable for some $b_0 \in B$, then it is finitely presented.*

Proof. Consider the maximal disk-hedgehog covering $p : E \to B$. 3.11 says its fibers are Baire spaces. Since they are countable, they must be discrete. Apply 7.6. $\square$

Let's turn to Problem 7.1. First, let us show that every small loop belongs to the kernel of $\pi_1(B, b_0) \to \pi(B, b_0)$. It shows that the hedgehog fundamental group eliminates some of the pathologies of the classical fundamental group.

Recall (see [21]) that a loop α at b_0 in B is called **small** if it can be homotoped relative to b_0 into any neighborhood U of b_0 in B.

Proposition 7.8. *Suppose B is path-connected. If $p : E \to B$ is a disk-hedgehog covering, then $[\alpha]$ is the neutral element of $\pi(p, b_0)$ for every small loop α at b_0.*

Proof. We may assume E is Peano by switching to its Peanification. Suppose α is a small loop at b_0 in B so that $[\alpha]$ is not the neutral element of $\pi(p, b_0)$. There is a lift $\tilde{\alpha}$ of α with $e_0 = \tilde{\alpha}(0) \neq e_1 = \tilde{\alpha}(1)$.

Choose a path-connected neighborhood U of b_0 in B such that the path component V of e_0 in $p^{-1}(U)$ is different from path-component W of e_1 in $p^{-1}(U)$. Suppose there is a loop β in U homotopic to α rel.b_0 in B. Its lift $\tilde{\beta}$ would join e_0 and e_1, a contradiction. □

Let's consider a more general question than 7.1: Characterize kernels of $\pi_1(B, b_0) \to \pi(p, b_0)$, where $p : E \to B$ is a disk hedgehog covering over a Peano space B.

As in [20, p.81], given an open cover $\mathcal{U}$ of X, $\pi(\mathcal{U}, x_0)$ is the subgroup of $\pi_1(X, x_0)$ generated by elements of the form $[\alpha * \gamma * \alpha^{-1}]$, where γ is a loop in some $U \in \mathcal{U}$ and α is a path from x_0 to $\gamma(0)$.

Lemma 7.9. *Suppose $P(B, b_0, \sim)$ has a whisker topology such that $\alpha \sim const$ implies $[\alpha] \in \pi(\mathcal{U}, b_0)$ for some open cover $\mathcal{U}$ of B. If $\beta(t)$, $t \in [0, 1]$, are paths in $P(B, b_0, \sim)$ forming a lift of a path γ starting at $[\alpha]$, then $\beta(t) * \gamma_t^{-1} * \alpha^{-1} \in \pi(\mathcal{U}, b_0)$ for all $t \in [0, 1]$.*

Proof. Let $S = \{t \in [0, 1] | \beta(t) * \gamma_t^{-1} * \alpha^{-1} \in \pi(\mathcal{U}, b_0)\}$. Clearly, $0 \in S$. It suffices to show that for any $t \in S$, $t < 1$, there is $s > t$ such that $[t, s] \subset S$ and that S contains its supremum. Given $t \in S$, $t < 1$, pick $V \in \mathcal{U}$ containing $\gamma(t)$ and choose a closed interval $W = [t, u]$ in $[0, 1]$, $u > t$, such that $\beta(s) \in N(\beta(s), V)$ for $s \in W$ and $\gamma(s) \in V$ for $s \in W$. Therefore, given $s \in W$, there is a path ω in V satisfying $\beta(s) \sim \beta(t) * \omega$. Notice ω joins $\gamma(t)$ and $\gamma(s)$.

The loop $\lambda = \omega * (\gamma|[t, s])^{-1}$ lies in V and $\beta(s) * \gamma_s^{-1} * \alpha^{-1} \sim \beta(t) * \omega * \gamma_s^{-1} * \alpha^{-1} \sim \beta(t) * \lambda * (\gamma|[t, s]) * \gamma_s^{-1} * \alpha^{-1} \sim \beta(t) * \lambda * \gamma_t^{-1} * \alpha^{-1} \sim (\beta(t) * \gamma_t^{-1} * \alpha^{-1}) * \alpha * \gamma_t * \lambda * \gamma_t^{-1} * \alpha^{-1}$ and the last loop belongs to $\pi(\mathcal{U}, b_0)$.

The same argument proves that the supremum of S belongs to S (we only used that s and t are sufficiently close). □

Proposition 7.10. *Let B be a Peano space. If $p : E \to B$ is a covering projection, then the kernel of $\pi_1(B, b_0) \to \pi(p, b_0)$ contains $\pi(\mathcal{U}, b_0)$, where $\mathcal{U}$ consists of all open subsets U of B that are evenly covered.*

Given an open cover $\mathcal{U}$ of B, the set of covering projections $q : E \to B$ for which each $U \in \mathcal{U}$ is evenly covered has a maximum p and the kernel of $\pi_1(B, b_0) \to \pi(p, b_0)$ is exactly $\pi(\mathcal{U}, b_0)$.

Proof. Obviously, elements of the form $[\alpha * \gamma * \alpha^{-1}]$, where γ is a loop in some $U \in \mathcal{U}$ and α is a path from b_0 to $\gamma(0)$ have a lift to E that is a loop, so they are trivial in $\pi(p, b_0)$.

Consider the end-point projection $p : P(B, b_0, \sim) \to B$ ($\alpha \sim \beta$ if and only if $[\alpha * \beta^{-1}] \in \pi(\mathcal{U}, b_0)$). It is a classical covering with each member of $\mathcal{U}$ being evenly covered (see [3] or use 7.9 to deduce it has unique path-lifting property and then construct sections over members of $\mathcal{U}$). Notice the kernel of $\pi_1(B, b_0) \to \pi(p, b_0)$ is exactly $\pi(\mathcal{U}, b_0)$. Indeed, if a loop γ in B lifts to a loop in $P(B, b_0, \sim)$, then 7.9 says the loop must belong to $\pi(\mathcal{U}, b_0)$.

Given any classical covering projection $q : E \to B$ with each member of $\mathcal{U}$ being evenly covered one can construct $f : P(B, b_0, \sim) \to E$ such that $q \circ f = p$ by lifting paths. That proves maximality of p. □

Definition 7.11. The intersection of all $\pi(\mathcal{U}, b_0)$, $\mathcal{U}$ ranging over all open covers of B, is called the **Spanier group** of (B, b_0) (see [10]).

By a **medium loop** we mean a loop α at b_0 that is not small and its homotopy class $[\alpha]$ belongs to the Spanier group. By a **big loop** we mean a loop α at b_0 that is neither medium nor small.

Proposition 7.12. *Let B be a Peano space. If p is the supremum of all classical coverings over B, then the kernel of $\pi_1(B,b_0) \to \pi(p,b_0)$ is exactly the Spanier group.*

Proof. Consider the end-point projection $p : P(B,b_0,\sim) \to B$ ($\alpha \sim \beta$ if and only if $[\alpha * \beta^{-1}] \in \pi(\mathcal{U},b_0)$ for all open covers $\mathcal{U}$ of B). Use 7.9 to deduce it has unique path-lifting property and then use 4.2 to show it is a disk-hedgehog covering. Notice the kernel of $\pi_1(B,b_0) \to \pi(p,b_0)$ is exactly the Spanier group. Indeed, if a loop γ in B lifts to a loop in $P(B,b_0,\sim)$, then 7.9 says the loop must belong to $\pi(\mathcal{U},b_0)$ for all open covers $\mathcal{U}$ of B.

Given any classical covering projection $q : E \to B$ with each member of $\mathcal{U}$ being evenly covered one can construct $f : P(B,b_0,\sim) \to E$ such that $q \circ f = p$ by lifting paths.

Suppose $q : E \to B$, E Peano, is a disk-hedgehog covering with $q(e) = b_0$ and maps $f_{\mathcal{U}} : E \to P(B,b_0,\sim_{\mathcal{U}})$ such that $f_{\mathcal{U}} \circ p_{\mathcal{U}} = q$ for each open cover $\mathcal{U}$ of B. Here $\alpha \sim_{\mathcal{U}} \beta$ if and only if $[\alpha * \beta^{-1}] \in \pi(\mathcal{U},b_0)$ and $p_{\mathcal{U}}$ is the end-point projection.

Given $x \in E$ and two path α_x, β_x from e to x, the loop $\alpha_x * \beta_x^{-1}$ must belong to the Spanier group as it can be factored through all $P(B,b_0,\sim_{\mathcal{U}})$, therefore the function $f(x) = [\gamma\alpha_x]$ (γ a fixed loop at b_0 in B) is well-defined and is continuous as p is an arc-hedgehog covering. As $p \circ f = q$, $q \geq p$. That proves maximality of p. $\square$

Corollary 7.13. *The the kernel of $\pi_1(B,b_0) \to \pi(B,b_0)$ contains all small loops and is contained in the union of small loops and medium loops.*

Let us show how direct wedge can be used to construct interesting spaces.

First of all, one can change the topology of the standard arc-hedgehog $\bigvee\limits_{n\in N} (I_n, 0_n)$ by requiring open neighborhoods of the base-point to contain all but finitely many 0_n's (instead of all but finitely many I_n's) and get a connected space that is not locally connected (a modified topologist's sine curve).

Second, one can change the topology of the standard disk-hedgehog $\bigvee\limits_{n\in N} (D^2_n, 0_n)$ by requiring open neighborhoods of the base-point to contain all but finitely many ∂D^2_n's (instead of all but finitely many D^2_n's) and get a space with properties similar to Harmonic Archipelago [2]: every loop is small.

It is easy to construct examples of medium loops by connecting two Harmonic Archipelagos by an arc. However, there is a more interesting example of Fischer-Zastrow [12] that can be used for that purpose. What is not clear is if that example does not become trivial once we kill all small loops.

Problem 7.14. *Construct a medium loop in a Peano space that does not belong to the normalizer of all small loops.*

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Cyclic and cocyclic maps and generalized Whitehead products

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Given co-H-spaces X and Y, B. Gray [13] has defined a co-H-space $X \circ Y$ and a natural transformation $X \circ Y \to X \vee Y$ which leads to a generalized Whitehead product. We make use of that product and sketch ideas on its dual to examine cyclic and cocyclic maps. Given spaces X and Y, some results on Gottlieb sets $\mathcal{G}(X,Y)$ and dual Gottlieb sets $\mathcal{DG}(X,Y)$ are stated.

Introduction

The Gottlieb group $G_n(X)$ of a space X is the subgroup of the homotopy group $\pi_n(X)$ of X consisting of homotopy classes of maps $f : \mathbb{S}^n \to X$ such that the map $f \vee \mathrm{id}_X : \mathbb{S}^n \vee X \to X$ admits an extension $F : \mathbb{S}^n \times X \to X$. The study of the properties and structure of the Gottlieb groups represents a fundamental problem in homotopy theory dating back to their introduction by D. Gottlieb in the 1960's

[8, 10]. Connections between the Gottlieb groups and fixed point theory [8, 15, 22], transformation groups [11, 20], covering spaces [11, 16] and the homotopy theory of fibrations [9, 12, 21] have been extensively researched.

The definition of $G_n(X)$ uses the concept of cyclic homotopies. K. Varadarajan [23] studies the role of cyclic and cocyclic (dual of cyclic) maps in the set-up of Eckmann-Hilton duality. The set of homotopy classes of cyclic maps $X \to Y$, denoted by $\mathcal{G}(X,Y)$ is a group provided X carries an H-cogroup structure. Dually, the set of homotopy classes of cocyclic maps $X \to Y$, denoted by $\mathcal{DG}(X,Y)$ is a group provided Y carries an H-group structure. Relationships between these generalized Gottlieb (dual Gottlieb groups) and the generalized Whitehead product (the dual generalized Whitehead product) [1] have been considered in [14, 17, 18, 19] and other various papers.

The aim of this paper is to present those results in the context of the so called Theriault product considered by B. Gray in [13] being an extended version of the generalized Whitehead product from [1] and its dual. The first section expounds the notions and clarify results needed in next two sections. Section 2 recalls results on cyclic maps and then takes up the systematic study of these maps in the context of results from [13].

Section 3 is devoted to cocyclic maps. First, their relations with the dual generalized Whitehead product [1] are summarized. In particular, a characterization of co-H-spaces in terms of the cocyclicity of maps is concluded. Then, following *mutatis mutandis* the construction presented by B. Gray in [13] and the cotelescope concept, we sketch ideas of the dual Theriault product extending the dual generalized Whitehead [1] and relate cocyclic maps to this product. Many results and proofs on the Theriault product can be dualized. The details will be published somewhere shortly.

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supporting during his stay.

2 Prerequisites

We concentrate with connected and based spaces having the homotopy type of CW-complexes. All maps and homotopies preserve base points. For simplicity, we sometimes use the same symbol for a map and its homotopy class. Denote by $[X, Y]$ the set of homotopy classes of continuous maps $X \to Y$ and write $\mathbb{S}^n$ for the n-dimensional sphere. In particular, let $\pi_n(X) = [\mathbb{S}^n, X]$ be the nth homotopy group of a space X for $n \geq 0$.

Next, write ΣX and ΩX for the suspension and the loop space of X. Recall that ΣX and ΩX are an H-cogroup and an H-group, respectively. If $f : X \to Y$ then for every space Z, we have homomorphisms $(\Sigma f)^* : [\Sigma Y, Z] \to [\Sigma X, Z]$ and $(\Omega f)_* : [Z, \Omega X] \to [Z, \Omega Y]$. Further, there are canonical natural maps $e : \Sigma\Omega X \to X$ and $e' : X \to \Omega\Sigma X$.

The following well-known results are frequently used:

Proposition 2.1. (1) *If X is a co-H-space, then there is a map $s : X \to \Sigma\Omega X$ such that $es \simeq \mathrm{id}_X$;*

(2) *If X is an H-space, then there is a map $s' : \Omega\Sigma X \to X$ such that $s'e' \simeq \mathrm{id}_X$;*

(3) *Let X and Y be an H-cogroup and an H-group, respectively. Then, $[X, Z]$ and $[Z, Y]$ are groups for any space Z.*

Let $X \flat Y$ be the flat product and $X \wedge Y$ the smash product, that is, the fibre and the cofibre of the inclusion $X \vee Y \hookrightarrow X \times Y$. Next, write $\Delta : X \to X \times X$ and $\nabla : X \vee X \to X$ for the diagonal and folding maps, respectively.

The Whitehead product $[-,-] : \pi_m(X) \times \pi_n(X) \to \pi_{m+n-1}(X)$, determined by the Whitehead map $w : \mathbb{S}^{m+n-1} \to \mathbb{S}^m \vee \mathbb{S}^n$ plays a crucial role in the homotopy theory. The generalized Whitehead map $w : \Sigma(X \wedge Y) \to \Sigma X \vee \Sigma Y$ constructed in [1] leads to the generalized Whitehead product

$$[-,-] : [\Sigma X, Z] \times [\Sigma Y, Z] \to [\Sigma(X \wedge Y), Z].$$

Now, let $\mathcal{CO}$ be the category of simply connected co-H-spaces and co-H-maps. In [13], a functor

$$\circ : \mathcal{CO} \times \mathcal{CO} \to \mathcal{CO}$$

(called the Theriault product) and a natural transformation $w : X \circ Y \to X \vee Y$ for co-H-spaces X, Y generalizing the Whitehead product have been defined. More precisely, in [13, Theorem 1, Theorem 2] it has been shown:

Theorem 2.2. *There is a functor*

$$\circ : \mathcal{CO} \times \mathcal{CO} \longrightarrow \mathcal{CO}$$

and equivalences in $\mathcal{CO}$:

(1) $(\Sigma X) \circ Y \cong X \wedge Y$;
(2) $\Sigma(X \circ Y) \cong X \wedge Y$;
(3) $(X_1 \vee X_2) \circ Y \cong (X_1 \circ Y) \vee (X_2 \circ Y)$

and homotopy equivalences:

(4) $X \circ Y \cong Y \circ X$;
(5) $(X \circ Y) \circ Z \cong X \circ (Y \circ Z)$.

Theorem 2.3. *There is a natural transformation*

$$w_\circ : X \circ Y \longrightarrow X \vee Y$$

which is the Whitehead product map in case X and Y are both suspensions. Furthermore, there is a homotopy equivalence

$$X \times Y \cong (X \vee Y) \cup_{w_\circ} C(X \circ Y),$$

where $(X \vee Y) \cup_{w_\circ} C(X \circ Y)$ is the mapping cone of $w_\circ : X \circ Y \longrightarrow X \vee Y$.

Notice that $w_\circ : X \circ Y \longrightarrow X \vee Y$ defines a map

$$[-,-]_\circ : [X, Z] \times [Y, Z] \to [X \circ Y, Z]$$

for any space Z.

3 Cyclic maps and evaluation groups

According to [23], a map $f : X \to Y$ is said to be *cyclic* if there exists a map $F : X \times Y \to Y$ such that the diagram

$$\begin{tikzcd}
X \vee Y \arrow[rr, "\nabla(f\vee \mathrm{id}_Y)"] \arrow[d, hook] & & Y \\
X \times Y \arrow[urr, "F"'] & &
\end{tikzcd}$$

is homotopy commutative.

Write $\mathcal{G}(X,Y)$ for the set of homotopy classes of cyclic maps from X to Y called the *Gottlieb subset* of $[X,Y]$. If X is an H-cogroup then by [23, Theorem 1.5] the subset $\mathcal{G}(X,Y) \subseteq [X,Y]$ is a subgroup of $[X,Y]$. If $X = \mathbb{S}^n$, the n-dimensional sphere then $\mathcal{G}(\mathbb{S}^n, Y) = G_n(Y)$ is called the nth *evaluation subgroup* of Y or the nth *Gottlieb group* defined in [8] for $n = 1$ and then in [10] for any $n \geq 1$. Then, $G_{n+k}(\mathbb{S}^n)$ and $G_{n+k}(\mathbb{F}P^n)$ have been extensively studied in [6] and [7], respectively, where $\mathbb{F}P^n$ is the projective space over $\mathbb{F}$ being the reals $\mathbb{R}$, complex numbers $\mathbb{C}$, quaternions $\mathbb{H}$ or the Cayley algebra $\mathbb{K}$.

To show the existence of cyclic maps, we recall:

Proposition 3.1 ([23, Lemmas 1.3 and 1.4])**.** *Let $f : X \to Y$ be a cyclic map and $g : Z \to X$ an arbitrary map. Then:*

(1) *$fg : Z \to Y$ is a cyclic map;*

(2) *if a map $g : Y \to Y'$ has a right homotopy inverse then $gf : X \to Y'$ is a cyclic map.*

In particular, let X be a co-H-space, $f : X \to Y$ and $e : \Sigma\Omega X \to X$ the usual map. Then f is cyclic if and only if $fe : \Sigma\Omega X \to Y$ is cyclic.

Proposition 3.2 ([17, Proposition 3.3])**.** *Let Y be a space. Then the following are equivalent:*

(1) *Y is an H-space;*

(2) id_Y *is cyclic;*

(3) *$\mathcal{G}(X,Y) = [X,Y]$ for any space X.*

Another way in which cyclic maps arise naturally is by fibrations. Suppose $F \to E \to B$ is a fibration. Then we have an operation $\rho : F \times \Omega B \to F$ and the restriction $\partial = \rho|_{\Omega B}$ is cyclic.

Now, we make use of Theorem 2.3 to deduce results being key ones in sequel.

Corollary 3.3. *Let X, Y be spaces. Then:*

(1) *the map $w_\circ : \Sigma\Omega X \circ \Sigma\Omega Y \to \Sigma\Omega X \vee \Sigma\Omega Y$ coincides with the generalized Whitehead map $w : \Sigma(\Omega X \wedge \Omega Y) \to \Sigma\Omega X \vee \Sigma\Omega Y$;*

(2) *there is the commutative diagram*

$$\begin{tikzcd}
X \circ Y \arrow[r, "w_\circ"] & X \vee Y \\
\Sigma\Omega X \circ \Sigma\Omega Y \arrow[u, "e \circ e"] \arrow[r, "w_\circ"] & \Sigma\Omega X \vee \Sigma\Omega Y. \arrow[u, "e \vee e"']
\end{tikzcd}$$

Then, the result [18, Proposition 4.6] leads to:

Proposition 3.4. *Let X be a co-H-space and $f : X \to Y$ a cyclic map. Then $[f, g]_\circ = 0$ for any map $g : Z \to Y$ provided Z is a co-H-space.*

Proof. Let $f : X \to Y$ be a cyclic map. Then by Proposition 3.1 the map $fe : \Sigma\Omega X \to Y$ is cyclic as well. Hence, in view of [18, Proposition 4.6], we get $[fe, ge] = 0$. Because X and Z are co-H-spaces, Corollary 3.3 leads to $[f, g]_\circ = 0$ and the proof is complete. □

Further [5, Proposition 2.3] and Proposition 2.1 yield:

Proposition 3.5. *For a map $f : X \to Y$ of H-groups, the following are equivalent:*

(1) *f_* maps $[Z, X]$ into the center of $[Z, Y]$;*

(2) *$\nabla(f \vee \mathrm{id}_Y)i \simeq \star$, where $i : X \flat Y \hookrightarrow X \vee Y$ is the inclusion map.*

If one of the conditions above is fulfilled, T. Ganea [5] says that f *maps X into the center of Y*.

The proof of the result below is a direct consequence of Corollary 3.3 and [14, Corollary 3].

Theorem 3.6. *Let X, Y be co-H-spaces and $f : X \to Y$. Then the following are equivalent:*

(1) *f is cyclic;*

(2) *f maps ΩX into the center of ΩY;*

(3) $[f, \mathrm{id}_Y]_\circ = 0$.

Theorem 3.6 generalized the result known to spheres: $f \in \mathcal{G}(\mathbb{S}^{n+k}, \mathbb{S}^n) = G_{n+k}(\mathbb{S}^n)$ *if and only if the Whitehead product* $[f, \mathrm{id}_{\mathbb{S}^n}] = 0$ which has been applied in [6] to find $G_{n+k}(\mathbb{S}^n)$ for $k \leq 13$. Certainly, the computations depend on the Whitehead product on spheres.

Now, let $i_1 : Y_1 \hookrightarrow Y_1 \vee Y_2$ and $i_2 : Y_2 \hookrightarrow Y_1 \vee Y_2$ be the inclusion maps. Then, Theorem 3.6 leads to the following generalization of [3, Proposition 2.3]:

Corollary 3.7. *Let X, Y_1, Y_2 be co-H-spaces and $f : X \to Y_1 \vee Y_2$. Then, f is cyclic if and only if $[f, i_1]_\circ = [f, i_2]_\circ = 0$.*

If A is an abelian group and $n \geq 2$ then the Moore space $M(A, n)$ is a co-H-space as a suspension of some space. Because $M(A_1 \oplus A_2, n) \cong M(A_1, n) \vee M(A_2, n)$ for some abelian groups A_1, A_2 [3, Proposition 2.3] has been applied to compute $G_n(M(A, n))$ provided A is a finitely generated abelian group. The paper [2] considers the set of homotopy classes of co-structures on a Moore space $M(A, n)$, where A is an abelian group and $n \geq 2$ is an integer. It is shown that for $n > 2$ the set has one element and for $n = 2$ the set is in one-to-one correspondence with $\mathrm{Ext}(A, A \otimes A)$. Further, a detailed investigation of the co-H-structures on $M(A, 2)$ in the case $A = \mathbb{Z}_m$, the integers mod m has been considered. It has been shown that all co-H-structures on $M(\mathbb{Z}_m, 2)$ are associative and commutative if m is odd, and all co-H-structures on $M(\mathbb{Z}_m, 2)$ are associative and non-commutative if m is even. Therefore, Corollary 3.7 should be useful to describe $G_2(M(A, 2))$ with respect to all possible co-H-structures on $M(A, 2)$ provided A is a finitely generated group or more generally, $A = \bigoplus_{i \in I} \mathbb{Z} \oplus \bigoplus_{j \in J} \mathbb{Z}_{m_j}$.

Let Y be an H-group and $f : X \to Y$. Recall that f is called *central* if $c(\mathrm{id}_Y \times f) \simeq \star$, where $c : Y \times Y \to Y$ is the basic commutator map. If

Y is an H-space then, in view of Proposition 2.1, the map $\Omega : [X,Y] \to [\Omega X, \Omega Y]$ given by $f \mapsto \Omega f$ is injective. Write $[\Omega X, \Omega Y]_{\mathcal{C}\Omega}$ for the subset of $[\Omega X, \Omega Y]$ consisting of those homotopy classes of maps Ωf which are central. Following [18, Definition 4.1], we set $\mathcal{C}(X,Y) = \Omega^{-1}[\Omega X, \Omega Y]_{\mathcal{C}\Omega}$. By [18, Propositions 4.6 and 5.1], it holds:

Proposition 3.8. *Let X, Y and Z be spaces.*

(1) *If $f \in \mathcal{C}(\Sigma X, Z)$ then $[f,g] = 0$ for any $g \in [\Sigma Y, Z]$.*

(2) *$\mathcal{C}(X,Y)$ is a subgroup contained in the center of $[X,Y]$ if X is a co-H-space with a right homotopy inverse and Y is any space.*

It follows that if X is a co-H-space with a right homotopy inverse, then for every space Y, $\mathcal{G}(X,Y) \subseteq \mathcal{C}(X,Y) \subseteq$ center of $[X,Y]$ as subgroups. In particular, $\mathcal{G}(X,Y)$ and $\mathcal{C}(X,Y)$ are abelian groups provided X is a co-H-space. This generalizes Gottlieb's result from [8] that the Gottlieb group $G_1(Y)$ lies in the center of the homotopy group $\pi_1(Y)$.

4 Cocyclic maps and coevaluation groups

According to [23], a map $f : X \to Y$ is said to be *cocyclic* if there is a map $F' : X \to X \vee Y$ such that the diagram

$$\begin{tikzcd} & X \times Y \\ X \arrow[ur, "(\mathrm{id}_X \times f)\Delta"] \arrow[r, "F'"'] & X \vee Y \arrow[u, hook] \end{tikzcd}$$

is homotopy commutative.

Write $\mathcal{DG}(X,Y)$ for the set of homotopy classes of cocyclic maps from X to Y called the *dual Gottlieb subset* of $[X,Y]$. If Y is an H-group then by [23, Theorem 1.5] the subset $\mathcal{DG}(X,Y) \subseteq [X,Y]$ is a subgroup of $[X,Y]$.

Certainly, every map $f : X \to Y$ is cocyclic provided X is a co-H-space.

Another way in which cocyclic maps arise naturally is by cofibrations (cf. [19]). Suppose $A \to B \to C$ is a cofibration. Then we have a cooperation $\phi : C \to C \vee \Sigma A$. Then the map $s = p_2\phi : C \to \Sigma A$ is cocyclic, where $p_2 : C \vee \Sigma A \to \Sigma A$ is the projection map.

Notice that if $f : X \to Y$ is a cocyclic map and $g : X' \to X$ has a left homotopy inverse then $fg : X' \to Y$ is also a cocyclic map. Then, in view of [23, Lemma 7.2], Proposition 3.1 can be dualized as follows:

Proposition 4.1. *Let $f : X \to Y$ be a cocyclic map. Then:*

(1) *$gf : X \to Z$ is a cocyclic map for an arbitrary map $g : Y \to Z$;*

(2) *if a map $g : X' \to X$ has a left homotopy inverse then $fg : X' \to Y$ is a cocyclic map.*

In particular, let Y be an H-space, $f : X \to Y$ and $e' : Y \to \Omega\Sigma Y$ the usual map. Then f is cocyclic if and only if $e'f : X \to \Omega\Sigma Y$ is cocyclic. Further, [19, Proposition 3.2] provides a characterization of a co-H-space in terms of the cocyclicity of maps.

Proposition 4.2. *Let X be a space. Then the following are equivalent:*

(1) *X is a co-H-space;*

(2) id_X *is cocyclic;*

(3) *$\mathcal{DG}(X, Y) = [X, Y]$ for any space Y.*

Recall from [1] that given spaces X and Y, there is a dual Whitehead map $w' : \Omega X \times \Omega Y \to \Omega(X \flat Y)$. This leads to the dual generalized Whitehead product

$$[-,-]' : [Z, \Omega X] \times [Z, \Omega Y] \to [Z, \Omega(X \flat Y)]$$

for any space Z.

Now, let $\mathcal{CO}'$ be the category of simply connected H-spaces and H-maps. Following *mutatis mutandis* the construction presented by B. Gray in [13] and the cotelescope construction, we get a functor

$$\circ' : \mathcal{CO}' \times \mathcal{CO}' \longrightarrow \mathcal{CO}'$$

(called the dual Theriault product) and a natural transformation

$$w' : X \times Y \longrightarrow X \circ' Y$$

which leads to a map

$$[-,-]_{\circ'} : [Z,X] \times [Z,Y] \to [Z, X \circ' Y]$$

for H-spaces X, Y and any space Z. Many results and proofs of $[-,-]_{\circ}$ can be dualized. We mention only that the products $[-,-]'$ and $[-,-]_{\circ'}$ coincide provided X, Y are loop spaces. However, many cannot since $[-,-]_{\circ'}$ is not precise a dual of $[-,-]_{\circ}$. The details and dual version of Theorem 2.2 and Theorem 2.3 will be published somewhere shortly.

The dual version of Corollary 3.3 and the result [18, Proposition 4.6] yield:

Proposition 4.3. *Let Y be an H-space and $f : X \to Y$ a cocyclic map. Then $[f,g]_{\circ'} = 0$ for any map $g : X \to Z$ provided Z is an H-space.*

>From this a dual version of Corollary 3.7 follows:

Corollary 4.4. *Let X_1, X_2, Y be H-spaces and $f : X_1 \times X_2 \to Y$. Then, f is cocyclic if and only if $[f,p_1]_{\circ'} = [f,p_2]_{\circ'} = 0$ for the projection maps $p_1 : X_1 \times X_2 \to X_1$ and $p_2 : X_1 \times X_2 \to X_2$.*

Let A be an abelian group and $n \geq 2$. Then the associated Eilenberg-MacLane space $K(A,n)$ inherits an H-structure. Because $K(A_1 \times A_2, n) \cong K(A_1,n) \times K(A_2,n)$ for any abelian groups A_1, A_2, Corollary 4.4 should be very useful to compute $\mathcal{DG}(K(A,n),Y)$ provided that A is an abelian finitely generated group and Y is an H-space.

The dual version of Proposition 3.5 and [5, Proposition 2.3] lead to:

Proposition 4.5. *For a map $f : X \to Y$ of H-cogroups, the following are equivalent:*

(1) *f^* maps $[Y,Z]$ into the center of $[X,Z]$;*

(2) *$j(\mathrm{id}_X \times f)\Delta \simeq \star$, where $j : X \times Y \to X \wedge Y$ is the quotient map.*

If one of the conditions above is fulfilled, we follow T. Ganea [5] to say that f *maps X into the cocenter of Y*. Let X be an H-cogroup and $f : X \to Y$. Recall that f is called *cocentral* if $(\mathrm{id}_X \vee f)c \simeq \star$, where $c : X \to X \vee X$ is the basic cocommutator map.

If X is a co-H-space then the map $\Sigma : [X,Y] \to [\Sigma X, \Sigma Y]$ given by $f \mapsto \Sigma f$ is injective. A subset $\mathcal{DC}(X,Y)$ of $[X,Y]$ which is the dual of $\mathcal{C}(X,Y)$ has been studied in [19]. If Y is an H-space then the map $\Sigma :$ $[X,Y] \to [\Sigma X, \Sigma Y]$ given by $f \mapsto \Sigma f$ is injective. Let $[\Sigma X, \Sigma Y]_{\mathcal{C}\Sigma}$ denote the subset of $[\Sigma X, \Sigma Y]$ consisting of those homotopy classes of maps Σf which are cocentral. Following [19, Definition 4.7], we set $\mathcal{DC}(X,Y) = \Sigma^{-1}[\Sigma X, \Sigma Y]_{\mathcal{C}\Sigma}$.

In view of [19, Propositions 4.8 and 5.2], it holds:

Proposition 4.6. *Let X, Y and Z be spaces.*

(1) *If $f \in \mathcal{DC}(Z, \Omega X)$ then $[f,g]' = 0$ for any $g \in [Z, \Omega Y]$;*

(2) *the set $\mathcal{DC}(X,Y)$ is a subgroup contained in the center of $[X,Y]$ if Y is an H-space with a left homotopy inverse and X is any space.*

It follows that if Y is an H-space with a right homotopy inverse, then for every space X there are inclusions $\mathcal{DG}(X,Y) \subseteq \mathcal{DC}(X,Y) \subseteq$ center of $[X,Y]$ of subgroups. In particular, $\mathcal{DG}(X,Y)$ and $\mathcal{DC}(X,Y)$ are abelian groups provided X is an H-space.

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Derivatives of skew-symmetric and symmetric vector-valued tensors

Second order elliptic operator of Laplace type on bundles of vector-valued tensors on a Lie algebroid are introduced and investigated. The Weitzenböeck type formulas in the case of skew-symmetric and symmetric tensors are derived.

1. Introduction

A Lie algebroid over a manifold M is a vector bundle A over M with a homomorphism of vector bundles $\varrho_A : A \to TM$ called an *anchor*, and a real Lie algebra structure $(\Gamma(A), [\![\cdot,\cdot]\!])$ such that $[\![a, fb]\!] = f[\![a, b]\!] + \varrho_A(a)(f) \cdot b$ for all $a, b \in \Gamma(A)$, $f \in \mathcal{C}^\infty(M)$. If the anchor is constant rank [surjective] we say that the Lie algebroid is *regular* [*transitive*]. Any smooth manifold M defines a Lie algebroid, where $A = TM$ with the identity anchor and the natural Lie algebra of vector fields on M. Other examples of Lie algebroids are: Lie algebras, integrable distributions (in particular foliations), cotangent bundles of Poisson manifolds, Lie algebroids of principal bundles.

For more complete treatment of the category of Lie algebroids and its connections we refer to: [9], [6], [10], [7], [1].

This article is an extension of our paper [3] where generalized gradients in the sense of Stein and Weiss on Lie algebroids were introduced and investigated. Stein-Weiss gradients are irreducible (with respect to the action of the orthogonal group) summands of a covariant derivative (cf. [14]). The exterior derivative on skew-symmetric forms and its coderivative, the Ahlfors operator ([13]) and in particular the Cauchy-Riemann operator are the examples. A connection in a Lie algebroid A has a natural extension to the first order linear operator

$$\nabla : \Gamma(\bigwedge^k A^*) \longrightarrow \Gamma(A^* \otimes \bigwedge^k A^*).$$

The last bundle has the following splitting onto three irreducible summands:

$$\Gamma(A^* \otimes \bigwedge^k A^*) = \Gamma(\bigwedge^{k+1} A^*) \oplus \Gamma(\bigwedge^{1,k} A^*) \oplus \Gamma(\bigwedge^{k-1} A^*)$$

(cf. [3]). So, generalized gradients in this case are compositions of ∇ with the projections defined by the splitting. Here, we are going to focus on two gradients: exterior derivative d^a and its conjugate d^{a*} acting on skew-symmetric tensors and being—up to multiplicative constants—compositions of ∇ with the projections on the first and on the third summand respectively. In the case of the bundle of symmetric forms an analogous splitting leads to their symmetric counterparts d^s, d^{s*} acting on symmetric tensors. In the both cases a proper composition, namely

$$\triangle^a = d^{a*} d^a + d^a d^{a*}$$

in the first case and

$$\Delta^s = d^{s*} d^s - d^s d^{s*}$$

in the other, lead to important second order differential operators. Both of them are elliptic and, like the Bochner Laplacian $\nabla^*\nabla$, are of metric symbol (see sections 3 and 4). As a consequence we derive Weitzenböck type formulas in each case:

$$\Delta = \nabla^*\nabla \pm \mathcal{R} \mp \mathcal{T} - \mathcal{M}$$

(cf. theorems 3 and 8). The formulas describe exact relations of Δ to the Bochner Laplacian. The relations depends explicitly on three indicators of the connection: its curvature (the operator $\mathcal{R}$), its torsion (the operator $\mathcal{T}$) and non-compatibility of the connection and the metric (the operator $\mathcal{M}$). It is important that the two second order linear elliptic operators differ practically by a tensor. In this context deriving its explicit shape seems to be essential.

In classical differential geometry the formula enables deriving many classical results establishing the relation between the topological structure of an algebroid and its geometry. By the standard Bochner technique, from the Weitzenböck formula, one can get then information on existence or nonexistence of some important deformations like isometric, projective, conformal (cf. [15] by K. Yano). One can also get some information on cohomologies (Betti numbers, [16]) or on lower bounds for spectrum of Δ (cf. [5]). Many possible applications of Weitzenböck types formulas can be found in the paper [4] by J.-P. Bourguignon.

It seems to be interesting that the two quiet antipodal cases: the skew-symmetric and the symmetric one behave so similar. To stress this harmony we apply exactly the same arrangement of the material in the both cases. In the case of a general Lie algebroid there is no equivalent of global (integral) scalar product even if the algebroid bundle carries a Riemannian structure. The adjoint operators are then defined here as the negative traces of suitable parts of the covariant derivative. They coincide then in the particular case of the algebroid of the tangent bundle of a compact Riemannian manifold with the operators adjoint with respect to global (integral) scalar product. In contrast to [3] we consider here the tensors (forms) with values in a given vector bundle. This bundle needs not to have any additional structure like algebraic or metric. It is equipped with a connection only.

2. The exterior covariant derivative for an arbitrary connection

Let $(A, \varrho_A, [\![\cdot,\cdot]\!])$ be a Lie algebroid over a manifold M and let E be a vector bundle over M. Let $\mathscr{A}(A,E) = \bigoplus_{p\geq 0} \mathscr{A}^k(A,E)$, where $\mathscr{A}^k(A,E) = \Gamma(\bigwedge^k A^* \otimes E)$, be the $\mathcal{C}^\infty(M)$-module of skew-symmetric forms on the Lie algebroid A of values in the vector bundle E. $\mathscr{A}(A,E)$ is the module over the ring $\mathcal{C}^\infty(M)$ and the module over the algebra $\mathscr{A}(A) = \mathscr{A}(A, M\times\mathbb{R})$ with the multiplication defined in the following way:

$$\wedge : \mathscr{A}^p(A, M\times\mathbb{R}) \times \mathscr{A}^q(A,E) \longrightarrow \mathscr{A}^{p+q}(A,E),$$

$$(\omega\wedge\eta)(a_1,\ldots,a_{p+q}) = \sum_{\sigma\in S(p,q)} \operatorname{sgn}\sigma\cdot\omega\left(a_{\sigma(1)},\ldots,a_{\sigma(p)}\right)\cdot\eta\left(a_{\sigma(p+1)},\ldots,a_{\sigma(p+q)}\right),$$

where $S(p,q)$ is the set of (p,q)-shuffles.

Let

$$\nabla : A \longrightarrow \mathcal{A}(E)$$

be an A-connection in E, i.e. a homomorphism of vector bundles A and $\mathcal{A}(E)$, which commutes with anchors, and where $\mathcal{A}(E)$ is the Lie algebroid of E. We recall (cf. [9]) that the module $\mathcal{CDO}(E)$ of sections of $\mathcal{A}(E)$ is the space of all covariant differential operators in E, i.e. $\mathbb{R}$-linear operators $\ell : \Gamma(E)\to\Gamma(E)$ such that there is $X_\ell\in\mathfrak{X}(M)$

satisfying $\ell(fe) = f\ell(e) + X_\ell(f)\, e$ for all $f \in \mathcal{C}^\infty(M)$ and $e \in \Gamma(E)$. ∇ defines a $\mathcal{C}^\infty(M)$-linear operator

$$\nabla : \Gamma(A) \longrightarrow \mathcal{CDO}(E)$$

of modules of sections which will be denoted also by ∇ and also called an A-*connection*. One can observe that

$$\operatorname{Sec} \varrho_{\mathcal{A}(E)} \circ \nabla = \operatorname{Sec} \varrho_A,$$

where $\operatorname{Sec} \varrho_{\mathcal{A}(E)}$ and $\operatorname{Sec} \varrho_A$ are morphisms of $\mathcal{C}^\infty(M)$-modules determined by the anchor $\varrho_{\mathcal{A}(E)}$ in the Lie algebroid $\mathcal{A}(E)$ and ϱ_A, respectively. The 2-form $\mathcal{R}^\nabla \in \mathscr{A}^2(A, \operatorname{End}(E))$ defined by

$$\mathcal{R}^\nabla(a,b) = \nabla_a \circ \nabla_b - \nabla_b \circ \nabla_a - \nabla_{[\![a,b]\!]}$$

is called the *curvature* of the A-connection ∇. We say that ∇ is *flat* if $\mathcal{R}^\nabla = 0$.

Recall that the exterior derivative $d^\nabla : \mathscr{A}^k(A,E) \to \mathscr{A}^{k+1}(A,E)$ determined by ∇ is defined by

$$\begin{aligned}(2.1)\qquad \left(d^\nabla \eta\right)(a_1,\dots,a_{k+1}) \;=\; &\sum_{j=1}^{k+1} (-1)^{j-1} \nabla_{a_j}\left(\eta\left(a_1,\dots\widehat{a}_j\dots,a_{k+1}\right)\right)\\ &+\sum_{i<j} (-1)^{i+j}\, \eta\left([\![a_i,a_j]\!], a_1,\dots\widehat{a}_i\dots\widehat{a}_j\dots,a_{k+1}\right).\end{aligned}$$

d^∇ is a first order differential operator giving a cohomology space if ∇ is flat. In particular, if ∇ is the anchor considered as an A-connection in the vector bundle $M \times \mathbb{R}$, $d^\nabla = d^{\varrho_A}$ gives the cohomology of the Lie algebroid A (cf. [11]).

Let ∇^A be an A-connection in A. By a *torsion* of ∇^A we mean the 2-form $T^A \in \mathscr{A}^2(A,A)$ given by

$$T^A(a,b) = \nabla^A_a b - \nabla^A_b a - [\![a,b]\!], \quad a,b \in \Gamma(A).$$

Denote the vector bundle $\bigotimes^k A^*$ by $A^{*\otimes k}$ and $\bigotimes A^* = \bigoplus_{k\ge 0} A^{*\otimes k}$ by $A^{*\otimes}$. ∇ and ∇^A induce an A-connection

$$\nabla : \Gamma(A) \longrightarrow \mathcal{CDO}\left(A^{*\otimes} \otimes E\right)$$

in the vector bundle $A^{*\otimes} \otimes E$ by

$$\left(\nabla_a \zeta\right)(a_1,\dots,a_p) = \nabla_a\left(\zeta(a_1,\dots,a_p)\right) - \sum_{j=1}^{p} \zeta\left(a_1,\dots,\nabla^A_a a_j,\dots,a_p\right),$$

$a, a_1, \dots, a_p \in \Gamma(A)$, $\zeta \in \Gamma(A^{*\otimes p} \otimes E)$.

The connection ∇ determines the differential operator

$$\nabla : \Gamma\left(A^{*\otimes p} \otimes E\right) \longrightarrow \Gamma\left(A^{*\otimes p+1} \otimes E\right)$$

given by

$$(2.2)\qquad (\nabla\zeta)(a_0,a_1,\dots,a_k) = \left(\nabla_{a_0}\zeta\right)(a_1,\dots,a_k)$$

for $\zeta \in \Gamma(A^{*\otimes p} \otimes E)$, $a_j \in \Gamma(A)$.

Let $a \in \Gamma(A)$. The substitution operator

$$i_a : \Gamma\left(A^{*\otimes} \otimes E\right) \longrightarrow \Gamma\left(A^{*\otimes} \otimes E\right)$$

on $\Gamma(A^{*\otimes} \otimes E)$ is defined by

$$(i_a\zeta)(a_1,\dots,a_{p-1}) = \zeta(a, a_1,\dots,a_{p-1})$$

for all $\zeta \in \Gamma(A^{*\otimes p} \otimes E)$, $a_1,\dots,a_{p-1} \in \Gamma(A)$.

Define the *second covariant derivative*

$$\nabla^2 = \nabla \circ \nabla : \Gamma\left(A^{*\otimes p} \otimes E\right) \longrightarrow \Gamma\left(A^{*\otimes p+2} \otimes E\right)$$

and for any $a, b \in \Gamma(A)$ the operator $\nabla^2_{a,b}$ such that

$$\nabla^2_{a,b} = i_a i_b \nabla^2,$$

i.e. $\nabla^2_{a,b}$ is a operator of the zero degree given explicitly by

$$\nabla^2_{a,b}\zeta = \nabla_a(\nabla_b \zeta) - \nabla_{\nabla^A_a b}\zeta \tag{2.3}$$

for $\zeta \in \Gamma(A^{*\otimes} \otimes E)$.

Lemma 1.

$$\nabla_a i_b = i_b \nabla_a + i_{\nabla^A_a b}$$

for any $a, b \in \Gamma(A)$.

Proof. Let $\theta \in \Gamma(A^{*\otimes p} \otimes E)$, $a_1, \ldots, a_p \in \Gamma(A)$. Then

$$\begin{aligned}
& (\nabla_a i_b \theta - i_b \nabla_a \theta)(a_1, \ldots, a_p) \\
= \; & (\nabla_a (i_b \theta))(a_1, \ldots, a_p) - (\nabla_a \theta)(b, a_1, \ldots, a_p) \\
= \; & (\nabla_a (\theta(b, a_1, \ldots, a_p))) - \sum_{s=1}^{p} \theta\left(b, a_1, \ldots, \nabla^A_a a_s, \ldots, a_p\right) \\
& - (\nabla_a (\theta(b, a_1, \ldots, a_p))) + \theta\left(\nabla^A_a b, a_1, \ldots, a_p\right) + \sum_{s=1}^{p} \theta\left(b, a_1, \ldots, \nabla^A_a a_s, \ldots, a_p\right) \\
= \; & \left(i_{\nabla^A_a b}\theta\right)(a_1, \ldots, a_p).
\end{aligned}$$

□

Lemma 2.

$$\mathcal{R}^{\nabla}_{a,b}\zeta = \nabla^2_{a,b}\zeta - \nabla^2_{b,a}\zeta + \nabla_{T^A(a,b)}\zeta$$

for $\zeta \in \Gamma(A^{*\otimes} \otimes E)$, $a, b \in \Gamma(A)$.

Proof. Use Lemma 1 to obtain:

$$\begin{aligned}
\nabla^2_{a,b}\zeta - \nabla^2_{b,a}\zeta &= i_b(\nabla_a(\nabla\zeta)) - i_a(\nabla_b(\nabla\zeta)) \\
&= \left(\nabla_a i_b - i_{\nabla^A_a b}\right)(\nabla\zeta) - \left(\nabla_b i_a - i_{\nabla^A_b a}\right)(\nabla\zeta) \\
&= \nabla_a(\nabla_b\zeta) - \nabla_{\nabla^A_a b}\zeta - \nabla_b(\nabla_a\zeta) + \nabla_{\nabla^A_b a}\zeta \\
&= \nabla_a(\nabla_b\zeta) - \nabla_b(\nabla_a\zeta) - \nabla_{[\![a,b]\!]}\zeta - \nabla_{\nabla^A_a b - \nabla^A_b a - [\![a,b]\!]}\zeta \\
&= \left(\mathcal{R}^{\nabla}_{a,b} - \nabla_{T^A(a,b)}\right)\zeta.
\end{aligned}$$

□

The curvature of $\nabla : \Gamma(A) \longrightarrow \mathcal{CDO}(A^{*\otimes} \otimes E)$ depends explicitly on curvatures of the connections $\nabla : \Gamma(A) \longrightarrow \mathcal{CDO}(E)$ and $\nabla^A : \Gamma(A) \longrightarrow \mathcal{CDO}(A)$.

Lemma 3. *If* $\eta \in \Gamma\left(A^{*\otimes k} \otimes E\right)$, $a, b, a_1, \ldots, a_k \in \Gamma(A)$, *then*

$$\left(\mathcal{R}^{\nabla}_{a,b}\eta\right)(a_1, \ldots, a_k) = \mathcal{R}^{\nabla}_{a,b}(\eta(a_1, \ldots, a_k)) - \sum_{s=1}^{k} \eta\left(a_1, \ldots, \mathcal{R}^{\nabla^A}_{a,b} a_s, \ldots a_k\right).$$

Proof. Let $\eta \in \Gamma\left(A^{*\otimes k} \otimes E\right)$, $a, b, a_1, \ldots, a_k \in \Gamma(A)$. Then

$$\begin{aligned}
&\left(\mathcal{R}^{\nabla}_{a,b}\eta\right)(a_1, \ldots, a_k) \\
= \;& \nabla_a\left(\nabla_b\left(\eta\left(a_1, \ldots, a_k\right)\right)\right) - \sum_{s=1}^{k} \nabla_a\left(\eta\left(a_1, \ldots, \nabla^A_b a_s, \ldots, a_k\right)\right) \\
& - \sum_{s=1}^{k} \nabla_b\left(\eta\left(a_1, \ldots, \nabla^A_a a_s, \ldots, a_k\right)\right) + \sum_{s=1}^{k} \eta\left(a_1, \ldots, \nabla^A_b\left(\nabla^A_a a_s\right), \ldots, a_k\right) \\
& + \sum_{s=1}^{k}\sum_{t\neq s} \eta\left(a_1, \ldots, \nabla^A_b a_t, \ldots, \nabla^A_a a_s \ldots, a_k\right) + \sum_{s=1}^{k} \nabla_a\left(\eta\left(a_1, \ldots, \nabla^A_b a_s, \ldots, a_k\right)\right) \\
& - \sum_{s=1}^{k}\sum_{t\neq s} \eta\left(a_1, \ldots, \nabla^A_b a_t, \ldots, \nabla^A_a a_s \ldots, a_k\right) - \sum_{s=1}^{k} \eta\left(a_1, \ldots, \nabla^A_a\left(\nabla^A_b a_s\right), \ldots, a_k\right) \\
& - \nabla_{[\![a,b]\!]}\left(\eta\left(a_1, \ldots, a_k\right)\right) + \sum_{s=1}^{k} \eta\left(a_1, \ldots, \nabla^A_{[\![a,b]\!]} a_s, \ldots, a_k\right).
\end{aligned}$$

Now, by collecting similar terms we obtain that

$$\begin{aligned}
&\left(\mathcal{R}^{\nabla}_{a,b}\eta\right)(a_1, \ldots, a_k) \\
= \;& \nabla_a\left(\left(\nabla_b\eta\right)(a_1, \ldots, a_k)\right) - \nabla_b\left(\left(\nabla_a\eta\right)(a_1, \ldots, a_k)\right) - \nabla_{[\![a,b]\!]}\left(\eta\left(a_1, \ldots, a_k\right)\right) \\
& - \sum_{s=1}^{k} \eta\left(a_1, \ldots, \nabla^A_a\left(\nabla^A_b a_s\right) - \nabla^A_b\left(\nabla^A_a a_s\right) - \nabla^A_{[\![a,b]\!]} a_s, \ldots a_k\right) \\
= \;& \mathcal{R}^{\nabla}_{a,b}\left(\eta\left(a_1, \ldots, a_k\right)\right) - \sum_{s=1}^{k} \eta\left(a_1, \ldots, \mathcal{R}^{\nabla^A}_{a,b} a_s, \ldots, a_k\right).
\end{aligned}$$

$\square$

Define the A-connection

$$\nabla : \Gamma(A) \longrightarrow \mathcal{CDO}\left(\bigwedge A^* \otimes E\right)$$

in the vector bundle $\bigwedge A^* \otimes E$ by

$$\left(\nabla_a\eta\right)(a_1, \ldots, a_p) = \nabla_a\left(\eta\left(a_1, \ldots, a_p\right)\right) - \sum_{j=1}^{p} \eta\left(a_1, \ldots, \nabla^A_a a_j, \ldots, a_p\right),$$

$a, a_1, \ldots, a_p \in \Gamma(A)$, $\eta \in \mathscr{A}^p(A, E)$. Observe that for all $\eta \in \mathscr{A}(A, E)$, $f \in \mathcal{C}^\infty(M) = \mathscr{A}^0(A, E)$, $a \in \Gamma(A)$ we have

$$\nabla_a(f \cdot \eta) = f \cdot \nabla_a\eta + \left(\varrho_A\right)_a(f) \cdot \eta, \tag{2.4}$$

where $\varrho_A : \Gamma(A) \longrightarrow \mathcal{CDO}\left(\bigwedge A^* \otimes (M \times \mathbb{R})\right)$ is the A-connection in the bundle $\bigwedge A^* \otimes (M \times \mathbb{R})$ determined by the pair of connections ϱ_A and ∇^A. So, we see that indeed, for every $a \in \Gamma(A)$, the operator ∇_a has values in $\mathcal{CDO}\left(\bigwedge A^* \otimes E\right)$.

Lemma 4. *If $\omega \in \mathscr{A}(A, M \times \mathbb{R})$, $\nu \in \Gamma(E)$, $a \in \Gamma(A)$, then*

$$\nabla_a(\omega \otimes \nu) = \left(\varrho_A\right)_a(\omega) \otimes \nu + \omega \otimes \nabla_a\nu. \tag{2.5}$$

Proof. Let $\nu \in \Gamma(E)$, $a \in \Gamma(A)$. If $\omega \in \mathscr{A}^0(A) = \mathcal{C}^\infty(M)$, (2.5) is equivalent to (2.4). Now, let $\omega \in \mathscr{A}^p(A)$, $a_1, \ldots, a_p \in \Gamma(A)$. Then:

$$\begin{aligned}
&\nabla_a(\omega \otimes \nu)(a_1, \ldots, a_p) \\
= \; &\nabla_a((\omega \otimes \nu)(a_1, \ldots, a_p)) - \sum_{j=1}^{p} (\omega \otimes \nu)\left(a_1, \ldots, \nabla_a^A a_j, \ldots, a_p\right) \\
= \; &\nabla_a(\omega(a_1, \ldots, a_p) \cdot \nu) - \sum_{j=1}^{p} \omega\left(a_1, \ldots, \nabla_a^A a_j, \ldots, a_p\right) \cdot \nu \\
= \; &\varrho_A(a)(\omega(a_1, \ldots, a_p)) \cdot \nu - \sum_{j=1}^{p} \omega\left(a_1, \ldots, \nabla_a^A a_j, \ldots, a_p\right) \cdot \nu + \omega(a_1, \ldots, a_p) \cdot \nabla_a(\nu) \\
= \; &\left((\varrho_A)_a(\omega) \otimes \nu + \omega \otimes \nabla_a \nu\right)(a_1, \ldots, a_p).
\end{aligned}$$

□

Lemma 5. *If $\omega \in \mathscr{A}(A)$, $\eta \in \mathscr{A}(A, E)$, $a \in \Gamma(A)$:*

$$\nabla_a(\omega \wedge \eta) = (\varrho_A)_a(\omega) \wedge \eta + \omega \wedge \nabla_a \eta.$$

Proof. Let $\omega \in \mathscr{A}^p(A)$, $\eta \in \mathscr{A}^q(A, E)$, $a \in \Gamma(A)$. Let η be a form $\eta' \otimes \nu$ for some $\eta' \in \mathscr{A}^q(A)$ and $\nu \in \Gamma(E)$. Lemma 4 implies that

$$\begin{aligned}
\nabla_a(\omega \wedge \eta) &= \nabla_a(\omega \wedge \eta' \otimes \nu) \\
&= (\varrho_A)_a(\omega \wedge \eta') \otimes \nu + (\omega \wedge \eta') \otimes \nabla_a \nu.
\end{aligned}$$

Since $(\varrho_A)_a$ is a differentiation in the algebra $\mathscr{A}(A)$, from Lemma 4 we obtain:

$$\begin{aligned}
\nabla_a(\omega \wedge \eta) &= ((\varrho_A)_a(\omega) \wedge \eta' + \omega \wedge (\varrho_A)_a(\eta')) \otimes \nu + (\omega \wedge \eta') \otimes \nabla_a \nu \\
&= (\varrho_A)_a(\omega) \wedge (\eta' \otimes \nu) + \omega \wedge ((\varrho_A)_a(\eta') \otimes \nu + \eta' \otimes \nabla_a \nu) \\
&= (\varrho_A)_a(\omega) \wedge \eta + \omega \wedge \nabla_a(\eta).
\end{aligned}$$

□

Now, define the operator $d^a : \mathscr{A}^k(A, E) \longrightarrow \mathscr{A}^{k+1}(A, E)$ by

$$d^a \eta = (k+1) \cdot \operatorname{Alt}(\nabla \eta), \tag{2.6}$$

where for any $\zeta \in \bigotimes^p A^*$ its *alternation* $\operatorname{Alt}\zeta$ is defined by

$$\operatorname{Alt}\zeta = \tfrac{1}{p!} \sum_{\sigma \in S_p} \operatorname{sgn}\sigma\,(\sigma\zeta).$$

So,

$$(d^a \eta)(a_1, \ldots, a_{k+1}) = \sum_{j=1}^{k+1} (-1)^{j-1} \left(\nabla_{a_j} \eta\right)(a_1, \ldots \widehat{a}_j \ldots, a_{k+1}), \tag{2.7}$$

where $\eta \in \mathscr{A}^k(A, E)$, $a_1, \ldots, a_{k+1} \in \Gamma(A)$. A relation between d and d^a describes the following

Lemma 6.

$$d^a = d^\nabla + d^T$$

where $d^T : \mathscr{A}^p(A, E) \longrightarrow \mathscr{A}^{p+1}(A, E)$ is the operator given by

$$\left(d^T \eta\right)(a_1, \ldots, a_{p+1}) = \sum_{i<j} (-1)^{i+j} \eta\left(T^A(a_i, a_j), a_1, \ldots \widehat{a}_i \ldots \widehat{a}_j \ldots, a_{p+1}\right)$$

for any $\eta \in \mathscr{A}^p(A, E)$, $a_1, \ldots, a_{p+1} \in \Gamma(A)$.

Proof. Let $\eta \in \mathscr{A}^k(A,E)$, $a_1,\dots,a_{p+1} \in \Gamma(A)$. Therefore

$$\begin{aligned}
&(\operatorname{Alt}(\nabla\eta))(a_1,\dots,a_{p+1})\\
=\ &\sum_{j=1}^{p+1}(-1)^{j-1}\left(\nabla_{a_j}\eta\right)(a_1,\dots\widehat{a}_j\dots,a_{p+1})\\
=\ &\sum_{j=1}^{p+1}(-1)^{j-1}\nabla_{a_j}\left(\eta(a_1,\dots\widehat{a}_j\dots,a_{p+1})\right)-\sum_{i<j}(-1)^{j-1}\eta\left(a_1,\dots,\nabla^A_{a_j}a_i,\dots\widehat{a}_j\dots,a_{p+1}\right)\\
&-\sum_{j<i}(-1)^{j-1}\eta\left(a_1,\dots\widehat{a}_j\dots,\nabla^A_{a_j}a_i,\dots,a_{p+1}\right)\\
=\ &\sum_{j=1}^{p+1}(-1)^{j-1}\nabla_{a_j}\left(\eta(a_1,\dots\widehat{a}_j\dots,a_{p+1})\right)\\
&+\sum_{i<j}(-1)^{i+j}\eta\left(\nabla^A_{a_i}a_j-\nabla^A_{a_j}a_i,a_1,\dots\widehat{a}_i\dots\widehat{a}_j\dots,a_{p+1}\right)\\
=\ &\sum_{j=1}^{p+1}(-1)^{j-1}\nabla_{a_j}\left(\eta(a_1,\dots\widehat{a}_j\dots,a_{p+1})\right)\\
&+\sum_{i<j}(-1)^{i+j}\eta\left([\![a_i,a_j]\!]+T^A(a_i,a_j),a_1,\dots\widehat{a}_i\dots\widehat{a}_j\dots,a_{p+1}\right)\\
=\ &\left(d^\nabla\eta\right)(a_1,\dots,a_{p+1})+\left(d^T\eta\right)(a_1,\dots,a_{p+1}).
\end{aligned}$$

□

Notice that if ∇^A is torsion-free, $d^a = d^\nabla$ (cf. also [2]).

3. Weitzenböck Formula for Skew-symmetric Forms

Assume that in the vector bundle A we have a Riemannian metric g. For any $k > 1$ and any $\zeta \in \Gamma(A^{*\otimes k}\otimes E)$ define the trace $\operatorname{tr}\zeta \in \Gamma(A^{*\otimes k-2}\otimes E)$ as the trace with respect to the first two arguments by

$$(\operatorname{tr}\zeta)(a_1,\dots,a_{k-2}) = \sum_{j=1}^{n}\zeta(e_j,e_j,a_1,\dots,a_{k-2}) \tag{3.1}$$

where $(e_1,\dots,e_n)$ is a local orthonormal frame of A ($n = \dim A_x$, $x \in M$). Define additionally $\operatorname{tr}\zeta = 0$ for $\zeta \in \Gamma(A^{*\otimes 1})$. One can see that tr do not depend on the choice of the frame.

By the *exterior coderivative* d^{a*} we mean the operator:

$$d^{a*} = -\operatorname{tr}\circ\nabla : \mathscr{A}^k(A,E) \longrightarrow \mathscr{A}^{k-1}(A,E). \tag{3.2}$$

Remark 1. In the case of invariantly oriented Lie algebroids we can use the integral fibre operator and a scalar product on the module $\mathscr{A}(A)$ such that d^{a*} is formally adjoint to $d^a = d^{\varrho_A}$ with respect to this product, see [8]. For a general Lie algebroid we do not have such a scalar product.

Define three differential operators of order zero. The first, a *Ricci type operator* $\mathcal{R}^a : \mathscr{A}(A,E) \to \mathscr{A}(A,E)$ defined by

$$(\mathcal{R}^a\eta)(a_1,\dots,a_k) = \sum_{j=1}^{n}\sum_{s=1}^{k}(-1)^{s-1}\left(\mathcal{R}^\nabla_{e_j,a_s}\eta\right)(e_j,a_1,\dots\widehat{a}_s\dots,a_k), \tag{3.3}$$

the operator $\mathcal{T}^a : \mathscr{A}(A,E) \to \mathscr{A}(A,E)$ by

$$(3.4)\qquad (\mathcal{T}^a\eta)(a_1,\ldots,a_k) = \sum_{j=1}^{n}\sum_{s=1}^{k}(-1)^{s-1}\left(\nabla_{T^A(e_j,a_s)}\eta\right)(a_1,\ldots\widehat{a}_s\ldots,a_k),$$

and next, the operator $\mathcal{M}^a : \mathscr{A}(A,E) \longrightarrow \mathscr{A}(A,E)$ by

$$(3.5)\quad (\mathcal{M}^a\eta)(a_1,\ldots,a_k) = \sum_{j=1}^{n}\sum_{s=1}^{k}(-1)^{s-1}\left(i_{\nabla^A_{a_s}e_j}i_{e_j} + i_{e_j}i_{\nabla^A_{a_s}e_j}\right)(\nabla\eta)(a_1,\ldots\widehat{a}_s\ldots,a_k),$$

where $\eta \in \mathscr{A}^k(A,E)$, $a_1,\ldots,a_k \in \Gamma(A)$, $(e_1,\ldots,e_n)$ is a local orthonormal frame of A, $\mathcal{R}^\nabla$ is the curvature tensor of the connection ∇. The first one $\mathcal{R}^a$ is the trace of the curvature tensor. The next $\mathcal{T}^a$ indicates a deviation of the connection from being torsion-free. The third $\mathcal{M}^a$ measures a non-compatibility of ∇ with the metric. By Lemma 2,

$$(3.6)\qquad \begin{aligned}&(\mathcal{R}^a\eta - \mathcal{T}^a\eta)(a_1,\ldots,a_k)\\ &= \sum_{j=1}^{n}\sum_{s=1}^{k}(-1)^{s-1}\left(\nabla^2_{e_j,a_s}\eta - \nabla^2_{a_s,e_j}\eta\right)(e_j,a_1,\ldots\widehat{a}_s\ldots,a_k).\end{aligned}$$

Moreover observe that the operators $\mathcal{R}^a$, $\mathcal{T}^a\eta$, $\mathcal{M}^a\eta$ can be written in the following forms

$$\begin{aligned}(\mathcal{R}^a\eta) &= \operatorname{Alt}\left(\sum_{j=1}^{n} i_{e_j}\left(\mathcal{R}^\nabla_{e_j,\cdot}\eta\right)\right),\\ \mathcal{T}^a\eta &= -\operatorname{Alt}\left(\sum_{j=1}^{n}\nabla_{T^{\nabla^A}(e_j,\cdot)}\eta\right),\\ \mathcal{M}^a\eta &= -\operatorname{Alt}\left(\sum_{j=1}^{n}\left(i_{\nabla^A e_j}i_{e_j} + i_{e_j}i_{\nabla^A e_j}\right)\right)(\nabla\eta).\end{aligned}$$

Define the *Laplace operator* on differential forms on the Lie algebroid A by

$$\Delta^a = d^{a*}d^a + d^a d^{a*}.$$

Recall that for a linear operator $P : \Gamma(F) \to \Gamma(F)$ of order m in a vector bundle F its symbol at a given point $x \in M$ is defined by

$$\sigma_P(e,\omega) = P(f^m\eta)(x)$$

for $e \in F_x$ and such $\omega \in A_x^*$ that $\omega = (df)(x)$ for some smooth function f with $f(x) = 0$, and where $\eta \in \Gamma(F)$, $\eta(x) = e$ (cf. [12]). The definition is independent either of f nor of η.

Observe that if A is transitive, Δ^a is a second order strongly elliptic operator with the metric symbol

$$\sigma_{\Delta^a}(\omega,\eta) = |\omega|^2\,\eta.$$

Indeed, let $x \in M$, $\omega \in A_x^*$, $e \in \Lambda^k A_x^* \otimes E_x$ and let $f \in \mathcal{C}^\infty(M)$, $s \in \Gamma\left(\Lambda^k A^* \otimes E\right)$ satisfy $f(x) = 0$, $(df)(x) = \omega$, $s(x) = e$. Then

$$\sigma_{d^a}(\omega,e) = d^a(fs)(x) = (d^a f \wedge s + f d^a s)(x) = \omega \wedge e.$$

Moreover, since $(\varrho_A)(f) = d^a f$, the relation (2.4) implies

$$\sigma_{d^{a*}}(\omega,e) = d^{a*}(fs)(x) = \left(i_{(df)^\sharp}s\right)(x) = i_{\omega^\sharp}e$$

where $\sharp : A^* \to A$ is the musical isomorphism determined by the metric g, i.e. for an 1-form $\xi \in \mathscr{A}^k(A, M\times\mathbb{R})$

$$g\left(\xi^\sharp,b\right) = i_b\xi \quad \text{for} \quad b \in \Gamma(A).$$

Hence

$$\sigma_{d^{a*}d^{a}}(\omega, e) = i_{\omega^\sharp}(\omega \wedge e) = i_{\omega^\sharp}\omega \wedge e - \omega \wedge i_{\omega^\sharp} e$$

and

$$\sigma_{d^{a}d^{a*}}(\omega, e) = \omega \wedge i_{\omega^\sharp} e.$$

Consequently,

$$\sigma_{\Delta^a}(\omega, e) = \sigma_{d^{a*}d^{a}+d^{a}d^{a*}}(\omega, e) = i_{\omega^\sharp}\omega \wedge e = g\left(\omega^\sharp, \omega^\sharp\right) e.$$

Now we write the explicit formulas for the two terms of Δ in the case of an arbitrary Lie algebroid A.

Theorem 1.

$$d^{a*}d^{a}\eta = -\operatorname{trace} \nabla^2 \eta + \sum_{j=1}^{n} \operatorname{Alt}\left(i_{e_j}\left(\nabla^2_{e_j,(\cdot)}\eta\right)\right)$$

for $\eta \in \mathscr{A}(A, E)$.

Proof. Let $\eta \in \mathscr{A}^k(A, E)$, $a_1, \ldots, a_k \in \Gamma(A)$ and $(e_1, \ldots, e_n)$ be a local orthonormal frame of A. By (2.7) and the definition of d^{a*} we obtain that

$$\begin{aligned}
&\left(d^{a*}d^{a}\eta\right)(a_1, \ldots, a_k) \\
&= -\sum_{j=1}^{n}\left(\nabla_{e_j}\left(d^{a}\eta\right)(e_j, a_1, \ldots, a_k)\right) + \sum_{j=1}^{n}\left(d^{a}\eta\right)\left(\nabla^A_{e_j}e_j, a_1, \ldots, a_k\right) \\
&+ \sum_{j=1}^{n}\sum_{s=1}^{k}\left(d^{a}\eta\right)\left(e_j, a_1, \ldots, \nabla^A_{e_j}a_s, \ldots, a_k\right) \\
&= -\sum_{j=1}^{n}\nabla_{e_j}\left(\left(\nabla_{e_j}\eta\right)(a_1, \ldots, a_k)\right) - \sum_{j=1}^{n}\sum_{s=1}^{k}(-1)^s\,\nabla_{e_j}\left(\left(\nabla_{a_s}\eta\right)(e_j, a_1, \ldots \widehat{a}_s \ldots, a_k)\right) \\
&+ \sum_{j=1}^{n}\left(\nabla_{\nabla^A_{e_j}e_j}\eta\right)(a_1, \ldots, a_k) + \sum_{j=1}^{n}\sum_{s=1}^{k}(-1)^s\left(\nabla_{a_s}\eta\right)\left(\nabla^A_{e_j}e_j, a_1, \ldots \widehat{a}_s \ldots, a_k\right) \\
&+ \sum_{j=1}^{n}\sum_{s=1}^{k}\left(\nabla_{e_j}\eta\right)\left(a_1, \ldots, \nabla^A_{e_j}a_s, \ldots, a_k\right) + \sum_{j=1}^{n}\sum_{s=1}^{k}(-1)^{s-1}\left(\nabla_{\nabla^A_{e_j}}\eta\right)(e_j, a_1, \ldots \widehat{a}_s \ldots, a_k) \\
&+ \sum_{j=1}^{n}\sum_{s=1}^{k}\sum_{s \neq t}(-1)^t\left(\nabla_{a_t}\eta\right)\left(e_j, a_1, \ldots \widehat{a}_t \ldots, \nabla^A_{e_j}a_s, \ldots, a_k\right)
\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{j=1}^{n}\left(\nabla_{e_j}\left(\nabla_{e_j}\eta\right)\right)(a_1,\dots,a_k) - \sum_{j=1}^{n}\sum_{s=1}^{k}\left(\nabla_{e_j}\eta\right)\left(a_1,\dots,\nabla^A_{e_j}a_s,\dots,a_k\right)\\
&- \sum_{j=1}^{n}\sum_{s=1}^{k}(-1)^s\left(\nabla_{e_j}\left(\nabla_{a_s}\eta\right)\right)(e_j,a_1,\dots\widehat{a}_s\dots,a_k)\\
&- \sum_{j=1}^{n}\sum_{s=1}^{k}(-1)^s\left(\nabla_{a_s}\eta\right)\left(\nabla^A_{e_j}e_j,a_1,\dots\widehat{a}_s\dots,a_k\right)\\
&- \sum_{j=1}^{n}\sum_{s=1}^{k}\sum_{s\neq t}(-1)^s\left(\nabla_{a_s}\eta\right)\left(e_j,a_1,\dots\widehat{a}_s\dots,\nabla^A_{e_j}a_s,\dots,a_k\right)\\
&+ \sum_{j=1}^{n}\left(\nabla_{\nabla^A_{e_j}e_j}\eta\right)(a_1,\dots,a_k) + \sum_{j=1}^{n}\sum_{s=1}^{k}(-1)^s\left(\nabla_{a_s}\eta\right)\left(\nabla^A_{e_j}e_j,a_1,\dots\widehat{a}_s\dots,a_k\right)\\
&+ \sum_{j=1}^{n}\sum_{s=1}^{k}\left(\nabla_{e_j}\eta\right)\left(a_1,\dots,\nabla^A_{e_j}a_s,\dots,a_k\right) + \sum_{j=1}^{n}\sum_{s=1}^{k}(-1)^{s-1}\left(\nabla_{\nabla^A_{e_j}}\eta\right)(e_j,a_1,\dots\widehat{a}_s\dots,a_k)\\
&+ \sum_{j=1}^{n}\sum_{s=1}^{k}\sum_{s\neq t}(-1)^t\left(\nabla_{a_t}\eta\right)\left(e_j,a_1,\dots\widehat{a}_s\dots,\nabla^A_{e_j}a_s,\dots,a_k\right)
\end{aligned}
$$

After collecting similar summands and using (2.3) one obtains

$$
\begin{aligned}
&\left(d^{a*}d^a\eta\right)(a_1,\dots,a_k)\\
&\qquad= -\sum_{j=1}^{n}\left(\nabla_{e_j}\left(\nabla_{e_j}\eta\right) - \nabla_{\nabla^A_{e_j}e_j}\eta\right)(a_1,\dots,a_k)\\
&\qquad\quad- \sum_{j=1}^{n}\sum_{s=1}^{k}(-1)^s\left(\nabla_{e_j}\left(\nabla_{a_s}\eta\right)\right)(e_j,a_1,\dots\widehat{a}_s\dots,a_k)\\
&\qquad\quad+ \sum_{j=1}^{n}\sum_{s=1}^{k}(-1)^{s-1}\left(\nabla_{\nabla^A_{e_j}}\eta\right)(e_j,a_1,\dots\widehat{a}_s\dots,a_k)\\
&\qquad= -\operatorname{trace}\nabla^2\eta\,(a_1,\dots,a_k) + \sum_{j=1}^{n}\sum_{s=1}^{k}(-1)^{s-1}\left(\nabla^2_{e_j,a_s}\eta\right)(e_j,a_1,\dots\widehat{a}_s\dots,a_k)\,.
\end{aligned}
$$

Moreover observe that

$$
\begin{aligned}
&\sum_{j=1}^{n}\sum_{s=1}^{k}(-1)^{s-1}\left(\nabla^2_{e_j,a_s}\eta\right)(e_j,a_1,\dots\widehat{a}_s\dots,a_k)\\
=\;&\sum_{j=1}^{n}\sum_{s=1}^{k}(-1)^{s-1}\left(i_{a_s}\left(i_{e_j}\nabla\left(\nabla\eta\right)\right)\right)(e_j,a_1,\dots\widehat{a}_s\dots,a_k)\\
=\;&\sum_{s=1}^{k}(-1)^{s-1}\left(\sum_{j=1}^{n}i_{e_j}i_{a_s}\left(i_{e_j}\nabla\left(\nabla\eta\right)\right)\right)(a_1,\dots\widehat{a}_s\dots,a_k)\\
=\;&\operatorname{Alt}\left(\sum_{j=1}^{n}i_{e_j}\left(\nabla^2_{e_j,(\cdot)}\eta\right)\right)(a_1,\dots,a_k)\,.
\end{aligned}
$$

□

Theorem 2.

$$
\begin{aligned}
& \left(d^{a}d^{a*}\eta\right)\left(a_{1},\ldots,a_{k}\right) \\
= \; & -\sum_{j=1}^{n}\sum_{s=1}^{k}\left(-1\right)^{s-1}\left(\nabla^{2}_{a_{s},e_{j}}\eta\right)\left(e_{j},a_{1},\ldots\widehat{a}_{s}\ldots,a_{k}\right) \\
& -\sum_{j=1}^{n}\sum_{s=1}^{k}\left(-1\right)^{s-1}\left(i_{\nabla^{A}_{a_{s}}e_{j}}i_{e_{j}}+i_{e_{j}}i_{\nabla^{A}_{a_{s}}e_{j}}\right)\left(\nabla\eta\right)\left(a_{1},\ldots\widehat{a}_{s}\ldots,a_{k}\right),
\end{aligned}
$$

i.e.

$$
d^{a}d^{a*}\eta=-\sum_{j=1}^{n}\operatorname{Alt}\left(i_{e_{j}}\left(\nabla^{2}_{(\cdot),e_{j}}\eta\right)\right)-\sum_{j=1}^{n}\operatorname{Alt}\left(i_{\nabla^{A}e_{j}}i_{e_{j}}+i_{e_{j}}i_{\nabla^{A}e_{j}}\right)\left(\nabla\eta\right)
$$

for $\eta\in\mathscr{A}\left(A,E\right)$, $a_{1},\ldots,a_{k}\in\Gamma\left(A\right)$.

Proof. Let $\eta\in\mathscr{A}^{k}\left(A,E\right)$, $a_{1},\ldots,a_{k}\in\Gamma\left(A\right)$ and $\left(e_{1},\ldots,e_{n}\right)$ be a local orthonormal frame of A. By (2.7) and the definition of d^{a*} we have

$$
\begin{aligned}
& \left(d^{a}d^{a*}\eta\right)\left(a_{1},\ldots,a_{k}\right) \\
= \; & \sum_{s=1}^{k}\left(-1\right)^{s-1}\left(\nabla_{a_{s}}\left(d^{*}\eta\right)\right)\left(a_{1},\ldots\widehat{a}_{s}\ldots,a_{k}\right) \\
= \; & -\sum_{s=1}^{k}\sum_{j=1}^{n}\left(-1\right)^{s-1}\left(\nabla_{a_{s}}\left(i_{e_{j}}\left(\nabla_{e_{j}}\eta\right)\right)\right)\left(a_{1},\ldots\widehat{a}_{s}\ldots,a_{k}\right) \\
= \; & -\sum_{s=1}^{k}\sum_{j=1}^{n}\left(-1\right)^{s-1}\nabla_{a_{s}}\left(\left(\nabla_{e_{j}}\eta\right)\left(e_{j},a_{1},\ldots\widehat{a}_{s}\ldots,a_{k}\right)\right) \\
& +\sum_{j=1}^{n}\sum_{s=1}^{k}\sum_{t\neq s}\left(-1\right)^{s-1}\left(\nabla_{e_{j}}\eta\right)\left(e_{j},a_{1},\ldots,\nabla^{A}_{a_{s}}a_{t},\ldots\widehat{a}_{s}\ldots,a_{k}\right) \\
= \; & -\sum_{s=1}^{k}\sum_{j=1}^{n}\left(-1\right)^{s-1}\left(\nabla_{a_{s}}\left(\nabla_{e_{j}}\eta\right)\right)\left(e_{j},a_{1},\ldots\widehat{a}_{s}\ldots,a_{k}\right) \\
& -\sum_{s=1}^{k}\sum_{j=1}^{n}\left(-1\right)^{s-1}\left(\nabla_{e_{j}}\eta\right)\left(\nabla^{A}_{a_{s}}e_{j},a_{1},\ldots\widehat{a}_{s}\ldots,a_{k}\right) \\
& -\sum_{s=1}^{k}\sum_{j=1}^{n}\sum_{t\neq s}\left(-1\right)^{s-1}\left(\nabla_{e_{j}}\eta\right)\left(e_{j},a_{1},\ldots,\nabla^{A}_{a_{s}}a_{t},\ldots\widehat{a}_{s}\ldots,a_{k}\right) \\
& +\sum_{j=1}^{n}\sum_{s=1}^{k}\sum_{t\neq s}\left(-1\right)^{s-1}\left(\nabla_{e_{j}}\eta\right)\left(e_{j},a_{1},\ldots,\nabla^{A}_{a_{s}}a_{t},\ldots\widehat{a}_{s}\ldots,a_{k}\right).
\end{aligned}
$$

Now, collecting similar terms one concludes that

$$
\begin{aligned}
& \left(d^a d^{a*} \eta\right)(a_1, \ldots, a_k) \\
= & -\sum_{s=1}^{k} \sum_{j=1}^{n} (-1)^{s-1} \left(\nabla_{a_s} \left(\nabla_{e_j} \eta\right)\right)(e_j, a_1, \ldots \widehat{a}_s \ldots, a_k) \\
& -\sum_{s=1}^{k} \sum_{j=1}^{n} (-1)^{s-1} \left(\nabla_{e_j} \eta\right)\left(\nabla^A_{a_s} e_j, a_1, \ldots \widehat{a}_s \ldots, a_k\right) \\
= & -\sum_{s=1}^{k} \sum_{j=1}^{n} (-1)^{s-1} \left(\nabla^2_{a_s, e_j} \eta\right)(e_j, a_1, \ldots \widehat{a}_s \ldots, a_k) \\
& -\sum_{s=1}^{k} \sum_{j=1}^{n} (-1)^{s-1} \left(\nabla_{\nabla^A_{a_s} e_j} \eta\right)(e_j, a_1, \ldots \widehat{a}_s \ldots, a_k) \\
& -\sum_{s=1}^{k} \sum_{j=1}^{n} (-1)^{s-1} \left(\nabla_{e_j} \eta\right)\left(\nabla^A_{a_s} e_j, a_1, \ldots \widehat{a}_s \ldots, a_k\right) \\
= & -\sum_{s=1}^{k} \sum_{j=1}^{n} (-1)^{s-1} \left(\nabla^2_{a_s, e_j} \eta\right)(e_j, a_1, \ldots \widehat{a}_s \ldots, a_k) \\
& -\sum_{s=1}^{k} \sum_{j=1}^{n} (-1)^{s-1} \left(i_{e_j} i_{\nabla^A_{a_s} e_j} + i_{\nabla^A_{a_s} e_j} i_{e_j}\right)(\nabla \eta)(a_1, \ldots \widehat{a}_s \ldots, a_k).
\end{aligned}
$$

□

As a consequence of theorems 1 and 2 we have the following

Theorem 3. (*Weitzenböck Formula for Skew-Symmetric Forms*)

$$\Delta^a = \nabla^* \nabla + \mathcal{R}^a - \mathcal{T}^a - \mathcal{M}^a \tag{3.7}$$

where $\mathcal{R}^a$, $\mathcal{T}^a$ *and* $\mathcal{M}^a$ *are the operators defined in* (3.3)—(3.5).

Observe that if there exists a local orthonormal frame of sections $(e_1, \ldots, e_n)$ with the property $\nabla^A_{e_i} e_j\big|_x = 0$ at a single point $x \in M$, then $\mathcal{M}^a$ is equal to zero. This condition is fulfilled in case $A = F \subset TM$ is an integrable distribution on M and ∇^A is the Levi-Civita connection. The assumption of existence of a local orthonormal frame of sections that have vanishing covariant derivatives at a single point implies that the isotropy algebra of A (i.e. $\ker \varrho_A|_x$) is abelian, and then $\mathcal{T}^a = 0$.

4. d^{a*} and Δ^a in the case of a metric connection

Consider some particular cases. Assume that ∇^A is metric (is compatible with g), i.e.

$$(\varrho_A \circ a)(g(b, c)) = g(\nabla_a b, c) + g(b, \nabla_a c) \quad \text{for all} \quad a, b, c \in \Gamma(A).$$

We see at once that then the operator $\mathcal{M}^a$ vanishes. Consequently, the Weitzenböck Formula reduces to the form

$$\Delta^a = \nabla^* \nabla + \mathcal{R}^a - \mathcal{T}^a.$$

If ∇ is a torsion-free A-connection on A, then $d^a = d^\nabla$ is the exterior derivative on A given in (2.1) and $\mathcal{T}^a = 0$. In particular, if $\nabla^A : \Gamma(A) \longrightarrow \mathcal{CDO}(A)$ is the Levi-Civita connection in A, i.e.

$$\begin{aligned} & 2g\left(\nabla^A_a b, c\right) \\ = \;& \left(\varrho_A \circ a\right)\left(g\left(b,c\right)\right) + \left(\varrho_A \circ b\right)\left(g\left(a,c\right)\right) - \left(\varrho_A \circ c\right)\left(g\left(a,b\right)\right) \\ & + g\left([\![a,b]\!],c\right) + g\left([\![c,b]\!],a\right) + g\left([\![c,a]\!],b\right) \end{aligned}$$

for any $a, b, c \in \Gamma(A)$ (then ∇^A is uniquely determined metric and torsion-free connection), the Laplacian reduces to its classical shape:

$$\Delta^a = \nabla^* \nabla + \mathcal{R}^a.$$

If ∇^A is metric, the coderivative d^{*a} we can expressed in the language of the Hodge stat operator.

Assume that A is oriented and let $\Omega \in \mathscr{A}^n(A, M \times \mathbb{R})$ be the volume form ($n = \dim A_x$, $x \in M$).

For any $a \in \Gamma(A)$ we will denote by a^* the 1-form dual to a with respect to g, i.e. $a^* = g(a, \cdot)$. We extend g to the scalar product $\langle \cdot, \cdot \rangle_g$ on $\mathscr{A}^k(A, M \times \mathbb{R})$ in the usual way putting

$$\langle a_1^* \wedge \ldots \wedge a_k^*, b_1^* \wedge \ldots b_k^* \rangle_g = \det\left(\langle a_i^*, b_j^* \rangle_g \right),$$

$a_1, \ldots, a_k, b_1, \ldots, b_k \in \Gamma(A)$.

Definition 1. *Let $(e_1, \ldots, e_n)$ be a local oriented orthonormal frame for A and $(e^{*1}, \ldots, e^{*n})$ — the dual local orthonormal frame for A^*. Let $I = (i_1, \ldots, i_p)$ and $J = (j_1, \ldots, j_{n-p})$, where $i_1 < \ldots < i_p$, $j_1 < \ldots < j_{n-p}$, be a complementary set such that (I, J) is a permutation of $\{1, \ldots, n\}$. Let*

$$\omega_I = e^{*i_1} \wedge \ldots \wedge e^{*i_p}, \quad \omega_J = e^{*j_1} \wedge \ldots \wedge e^{*j_{n-p}}, \quad \nu \in \Gamma(E).$$

Define a $\mathcal{C}^\infty(M)$-linear operator

$$* : \mathscr{A}^p(A, E) \longrightarrow \mathscr{A}^{n-p}(A, E)$$

by

$$*(\omega_I \otimes \nu) = \epsilon(I, J)\, \omega_J \otimes \nu,$$

where $\epsilon(I, J)$ is the sign of the permutation $(I, J) = (i_1, \ldots, i_p, j_1, \ldots, j_{n-p})$.

One can check that

$$\Omega \otimes (*\eta)(a_1, ..., a_{n-p}) = (-1)^{p(n-p)}\, a_1^* \wedge ... \wedge a_{n-p}^* \wedge \eta,$$

for any $a_1, ..., a_{n-p} \in \Gamma(A)$, $\eta \in \mathscr{A}^p(A, E)$.

Consequently, by properties of the star operator on scalar forms (cf. [2]) we obtain

Lemma 7. *For any $\nu \in \Gamma(E)$, $f \in \mathcal{C}^\infty(M)$, $\eta \in \mathscr{A}^p(A, E)$, $a, a_1, ..., a_{n-p+1} \in \Gamma(A)$ the following equalities are fulfilled:*

(a) $*(\Omega \otimes \nu) = \nu$, $*(f\Omega \otimes \nu) = f\nu$, $*(\nu) = \Omega \otimes \nu$,
(b) $(*\eta)(a_1, ..., a_{n-p}) = (-1)^{p(n-p)} * \left(a_1^* \wedge ... \wedge a_{n-p}^* \wedge \eta\right)$,
(c) $i_a(*\eta) = (-1)^p * (a^* \wedge \eta)$,
(d) $**\eta = (-1)^{p(n-p)}\eta$.

Now we are going to show that $*$ and a metric connection ∇ commute.

Theorem 4. *If ∇^A is a metric connection,*

$$*(\nabla_a \eta) = \nabla_a(*\eta) \tag{4.1}$$

for all $\eta \in \mathscr{A}(A, E)$, $a \in \Gamma(A)$.

Proof. Let $a \in \Gamma(A)$, $\omega \in \mathscr{A}^p(A, M \times \mathbb{R})$, $\nu \in E$. From Theorem 3.2 [2] we have

$$(\varrho_A)_a(*\omega) = *((\varrho_A)_a \omega).$$

Therefore, by (2.5) we obtain

$$\begin{aligned}
\nabla_a(*(\omega \otimes \nu)) &= \nabla_a(*\omega \otimes \nu) \\
&= (\varrho_A)_a(*\omega) \otimes \nu + (*\omega) \otimes \nabla_a \nu \\
&= *((\varrho_A)_a \omega) \otimes \nu + (*\omega) \otimes \nabla_a \nu \\
&= *((\varrho_A)_a \omega \otimes \nu + \omega \otimes \nabla_a \nu) \\
&= *(\nabla_a(\omega \otimes \nu)).
\end{aligned}$$

□

Lemma 8. *If $(e_1, \ldots, e_n)$ is a local frame of A and $(e_1^*, \ldots, e_n^*)$ is the dual local frame of A^*, then*

$$d^a \eta = \sum_{s=1}^{n} e_s^* \wedge (\nabla_{e_s} \eta)$$

for $\eta \in \mathscr{A}(A, E)$.

Proof. Let $\eta \in \mathscr{A}^k(A, E)$, $a_1, \ldots, a_{k+1} \in \Gamma(A)$. Then

$$\begin{aligned}
(d^a \eta)(a_1, \ldots, a_{k+1}) &= \sum_{j=1}^{k+1} (-1)^{j-1} \left(\nabla_{a_j} \eta\right)(a_1, \ldots \widehat{a}_j \ldots, a_{k+1}) \\
&= \sum_{\sigma \in S(1,p)} \operatorname{sgn} \sigma \left(\nabla_{a_{\sigma(1)}} \eta\right) \left(a_{\sigma(2)}, \ldots, a_{\sigma(k+1)}\right) \\
&= \sum_{\sigma \in S(1,p)} \operatorname{sgn} \sigma \left(\nabla_{\sum_{s=1}^{n} g\left(e_s, a_{\sigma(1)}\right) e_s} \eta\right) \left(a_{\sigma(2)}, \ldots, a_{\sigma(k+1)}\right) \\
&= \sum_{s=1}^{n} \sum_{\sigma \in S(1,p)} \operatorname{sgn} \sigma \, e_s^* \left(a_{\sigma(1)}\right) (\nabla_{e_s} \eta) \left(a_{\sigma(2)}, \ldots, a_{\sigma(k+1)}\right) \\
&= \left(\sum_{s=1}^{n} e_s^* \wedge \nabla_{e_s}(\eta)\right)(a_1, \ldots, a_{k+1}).
\end{aligned}$$

□

As a conclusion from lemmas 8, 7 (e) and 7 (c) we obtain the following expression of the exterior coderivative.

Theorem 5. *If ∇^A is a metric connection,*

$$d^{a*} \eta = (-1)^{n(p+1)+1} * d^a * \eta \tag{4.2}$$

for $\eta \in \mathscr{A}^p(A, E)$.

As a conclusion we obtain

Corollary 1. *If ∇^A is metric, then $d^a(\omega \wedge *\eta) = (d^{\varrho_A} \omega) \wedge *\eta + (-1)^{m+p} \omega \wedge (*d^{a*} \eta)$ for $\omega \in \mathscr{A}^m(A)$, $\eta \in \mathscr{A}^p(A, E)$.*

Proof. Observe

$$\begin{aligned}
\omega \wedge (*d^{*a}\eta) &= (-1)^{n(p+1)+1}\,\omega \wedge (*\,*\,d^a * \eta) \\
&= (-1)^{n(p+1)+1}\,\omega \wedge \left((-1)^{(n-p+1)(n-(n-p+1))}\,d^a\,(*\eta)\right) \\
&= (-1)^{n(p+1)+1}\,(-1)^{(n-p+1)(p-1)}\,\omega \wedge (d^a\,(*\eta)) \\
&= (-1)^{np+n+1+np-n+p(-p+1)+p-1}\,\omega \wedge (d^a\,(*\eta)) \\
&= (-1)^p\,\omega \wedge (d^a\,(*\eta))\,.
\end{aligned}$$

Hence

$$\begin{aligned}
d^a\,(\omega \wedge *\eta) &= (d^{\varrho_A}\omega) \wedge *\eta + (-1)^m\,\omega \wedge d^a\,(*\eta) \\
&= (d^{\varrho_A}\omega) \wedge *\eta + (-1)^{m+p}\,\omega \wedge (*d^{a*}\eta)\,.
\end{aligned}$$

□

5. Weitzenböck Formula for Symmetric Forms

Let $\mathscr{S}^k(A,E)$ be the $\mathcal{C}^\infty(M)$-module of all symmetric differential forms of values in the vector bundle E, i.e. the module of sections of $\mathsf{S}^k A^* \otimes E \subset A^{*\otimes k} \otimes E$ and $\mathscr{S}(A,E) = \bigoplus_{k\geq 0} \mathscr{S}^k(A,E)$.

Define the A-connection

$$\nabla : \Gamma(A) \longrightarrow \mathcal{CDO}(\mathsf{S}A^* \otimes E)$$

in the vector bundle $\mathsf{S}A^* \otimes E$ by

$$(\nabla_a\zeta)(a_1,\ldots,a_p) = \nabla_a(\zeta(a_1,\ldots,a_p)) - \sum_{j=1}^{p} \zeta\left(a_1,\ldots,\nabla^A_a a_j,\ldots,a_p\right), \tag{5.1}$$

$a, a_1, \ldots, a_p \in \Gamma(A)$, $\zeta \in \mathscr{S}^p(A,E)$. Observe that—like in the skew-symmetric case—we have

$$\nabla_a(f\cdot\zeta) = f\cdot\nabla_a\zeta + (\varrho_A)_a(f)\cdot\zeta \tag{5.2}$$

for all $\zeta \in \mathscr{S}(A,E)$, $f \in \mathcal{C}^\infty(M) = \mathscr{S}^0(A,E)$, $a \in \Gamma(A)$, where (ϱ_A) denote here the A-connection in $\mathsf{S}A^* \otimes (M\times\mathbb{R})$ determined by the pair of connections ϱ_A and ∇^A. So, indeed the operator ∇_a has values in $\mathcal{CDO}(\mathsf{S}A^* \otimes E)$ for every $a \in \Gamma(A)$. Moreover, if $\lambda \in \mathscr{S}(A, M\times\mathbb{R})$, $\nu \in \Gamma(E)$, $a \in \Gamma(A)$, then

$$\nabla_a(\lambda\otimes\nu) = ((\varrho_A)_a\lambda)\otimes\nu + \lambda\otimes\nabla_a\nu. \tag{5.3}$$

The $C^\infty(M)$-module $\mathscr{S}(A,E)$ is equipped with the structure of the module over the algebra $\mathscr{S}(A, M\times\mathbb{R})$ with the multiplication

$$\odot : \mathscr{S}^p(A, M\times\mathbb{R}) \times \mathscr{S}^q(A,E) \longrightarrow \mathscr{S}^{p+q}(A,E)$$

defined by

$$(\lambda\odot\zeta)(a_1,\ldots,a_{p+q}) = \sum_{\sigma\in S(p,q)} \lambda\left(a_{\sigma(1)},\ldots,a_{\sigma(p)}\right)\cdot\zeta\left(a_{\sigma(p+1)},\ldots,a_{\sigma(p+q)}\right).$$

Observe that if $\lambda \in \mathscr{S}(A, M\times\mathbb{R})$, $\zeta \in \mathscr{S}(A,E)$, $a \in \Gamma(A)$:

$$\nabla_a(\lambda\odot\zeta) = ((\varrho_A)_a\lambda)\odot\eta + \lambda\odot(\nabla_a\zeta)\,.$$

Define the *symmetric derivative* $d^s : \mathscr{S}^k \longrightarrow \mathscr{S}^{k+1}$ by

$$(d^s\eta)(a_1,\ldots,a_{k+1}) = \sum_{j=1}^{k+1} \left(\nabla_{a_j}\eta\right)(a_1,\ldots\widehat{a}_j\ldots,a_{k+1}) \tag{5.4}$$

for $\eta \in \mathscr{S}^k$, $a_1, \dots, a_{k+1} \in \Gamma(A)$.

One can observe that

$$d^s = (k+1) \cdot (\mathrm{Sym} \circ \nabla) \quad \text{on} \quad \mathscr{S}^k(A, E) \tag{5.5}$$

where Sym is the *symmetrizer* given by

$$(\mathrm{Sym}\,\vartheta)(a_1, \dots, a_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \vartheta\left(a_{\sigma(1)}, \dots, a_{\sigma(k)}\right) \text{ for all } \vartheta \in \Gamma\left(A^{*\otimes k} \otimes E\right).$$

By the *symmetric coderivative* d^{s*} we mean the operator

$$d^{s*} = -\operatorname{tr} \circ \nabla|_{\mathscr{S}^k(A,E)} : \mathscr{S}^k(A, E) \longrightarrow \mathscr{S}^{k-1}(A, E) \tag{5.6}$$

where $\nabla : \Gamma\left(A^{*\otimes k} \otimes E\right) \longrightarrow \Gamma\left(A^{*\otimes k+1} \otimes E\right)$ is defined in (2.2), i.e. explicitly

$$(d^{s*}\zeta)(a_1, \dots, a_{k-2}) = \sum_{j=1}^{n} \zeta(e_j, e_j, a_1, \dots, a_{k-2})$$

for $\zeta \in \mathscr{S}^k(A, E)$, $a_1, \dots, a_{k-2} \in \Gamma(A)$.

Define the Laplace-type operator on symmetric tensors by

$$\Delta^s = d^{s*}d^s - d^s d^{s*}.$$

Example 1. Consider the Lie algebroid $A = T\mathbb{R}^n$ and the trivial bundle $E = M \times \mathbb{R}$. Take

$$\omega = \sum_{|\alpha|=k} \omega_\alpha dx_1^{\alpha_1} \odot dx_2^{\alpha_2} \odot \cdots \odot dx_n^{\alpha_n} \in \mathscr{S}^k(A, M \times \mathbb{R})$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, $\omega_\alpha \in C^\infty(M)$. Observe that

$$\nabla\omega = \sum_{j=1}^{n} \sum_{|\alpha|=k} \frac{\partial \omega_\alpha}{\partial x_j} dx_j \otimes dx_1^{\alpha_1} \odot dx_2^{\alpha_2} \odot \cdots \odot dx_n^{\alpha_n}.$$

and

$$\begin{aligned} d^s\omega &= \sum_{j=1}^{n} \sum_{|\alpha|=k} \frac{\partial \omega_\alpha}{\partial x_j} dx_j \odot dx_1^{\alpha_1} \odot dx_2^{\alpha_2} \odot \cdots \odot dx_n^{\alpha_n} \\ &= \sum_{j=1}^{n} \sum_{|\alpha|=k} \frac{\partial \omega_\alpha}{\partial x_j} dx_1^{\alpha_1} \odot dx_2^{\alpha_2} \odot \cdots \odot dx_j^{\alpha_j+1} \odot \cdots \odot dx_n^{\alpha_n} \end{aligned}$$

So,

$$\begin{aligned} d^{s*}\omega &= -\operatorname{tr} \nabla\omega \\ &= \sum_{s=1}^{n} i_{e_s} \left(\sum_{j=1}^{n} \sum_{|\alpha|=k} \frac{\partial \omega_\alpha}{\partial x_j} \delta_j^s \otimes dx_1^{\alpha_1} \odot \cdots \odot \alpha_s dx_s^{\alpha_s - 1} \odot \cdots \odot dx_n^{\alpha_n} \right) \\ &= \sum_{j=1}^{n} \sum_{|\alpha|=k} \frac{\partial \omega_\alpha}{\partial x_j} \alpha_j dx_1^{\alpha_1} \odot \cdots \odot dx_j^{\alpha_j - 1} \odot \cdots \odot dx_n^{\alpha_n}. \end{aligned}$$

Consequently,

$$\begin{aligned}\Delta^s\omega &= d^{s*}d^s\omega - d^s d^{s*}\omega \\ &= -\sum_{|\alpha|=k}\frac{\partial^2\omega_\alpha}{\partial x_j^2}\, dx_1^{\alpha_1}\odot dx_2^{\alpha_2}\odot\cdots\odot dx_n^{\alpha_n} \\ &= -\sum_{|\alpha|=k}\left(\Delta^s\omega_\alpha\right)\, dx_1^{\alpha_1}\odot dx_2^{\alpha_2}\odot\cdots\odot dx_n^{\alpha_n}\end{aligned}$$

where $\Delta^s\omega_\alpha = \Delta^a\omega_\alpha$ is the classical Laplacian on the smooth function ω_α.

Notice that if A is transitive, Δ^s is a second order strongly elliptic operator with the metric symbol

$$\sigma_{\Delta^s}(\omega,\eta) = |\omega|^2\eta, \quad \omega\in\mathscr{S}^k(A, M\times\mathbb{R}),\ \eta\in\mathscr{S}^k(A,E).$$

Indeed, take $x\in M$, $e\in\mathsf{S}^k A_x^*\otimes E_x$, $\zeta\in\mathscr{S}^k(A,E)$ and $\omega\in A_x^*$ such that $\omega=(df)(x)$ for some smooth function f satisfying $f(x)=0$ and $\zeta(x)=e$. Since $(\varrho_A)(f)=d^s f = d^a f$, the relation (5.2) implies that

$$\sigma_{d^s}(\omega,e) = d^s(f\zeta)(x) = \left(d^s f\odot\zeta + f d^s\zeta\right)(x) = \omega\odot e$$

and

$$\sigma_{d^{s*}}(\omega,e) = d^{s*}(f\zeta)(x) = \left(i_{(df)^\sharp}\zeta\right)(x) = i_{\omega^\sharp}e;$$

hence

$$\sigma_{d^{s*}d^s}(\omega,e) = i_{\omega^\sharp}(\omega\odot e) = i_{\omega^\sharp}\omega\odot e + \omega\odot i_{\omega^\sharp}e$$

and

$$\sigma_{d^s d^{s*}}(\omega,e) = \omega\odot i_{\omega^\sharp}e.$$

Consequently,

$$\sigma_{\Delta^s}(\omega,e) = \sigma_{d^{s*}d^s + d^s d^{s*}}(\omega,e) = i_{\omega^\sharp}\omega\odot e = g\left(\omega^\sharp,\omega^\sharp\right)e.$$

Define the *symmetric Ricci type operator*

$$\mathcal{R}^s : \mathscr{S}(A,E)\longrightarrow\mathscr{S}(A,E)$$

by

$$\left(\mathcal{R}^s\zeta\right)(a_1,\ldots,a_k) = \sum_{j=1}^n\sum_{s=1}^k\left(\mathcal{R}^\nabla_{e_j,a_s}\zeta\right)(e_j,a_1,\ldots\widehat{a}_s\ldots,a_k),$$

the operator

$$\mathcal{T}^s : \mathscr{S}(A,E)\longrightarrow\mathscr{S}(A,E)$$

by

$$\left(\mathcal{T}^s\zeta\right)(a_1,\ldots,a_k) = \sum_{j=1}^n\left(\nabla_{T^A(e_j,a_s)}\zeta\right)(a_1,\ldots,\widehat{a}_s,\ldots,a_k),$$

and next,

$$\mathcal{M}^s : \mathscr{S}(A,E)\longrightarrow\mathscr{S}(A,E)$$

by

$$\left(\mathcal{M}^s\zeta\right)(a_1,\ldots,a_k) = \sum_{j=1}^n\sum_{s=1}^k\left(i_{\nabla^A_{a_s}e_j}i_{e_j} + i_{e_j}i_{\nabla^A_{a_s}e_j}\right)(\nabla\zeta)(a_1,\ldots\widehat{a}_s\ldots,a_k),$$

where $\zeta \in \mathscr{S}^k(A,E)$, $a_1,\dots,a_k \in \Gamma(A)$, $(e_1,\dots,e_n)$ is a local orthonormal frame of A, $\mathcal{R}^\nabla$ is the curvature tensor of the connection $\nabla : \Gamma(A) \to \mathcal{CDO}(\mathsf{S}^k A^* \otimes E)$ defined in (5.1). Hence, by Lemma 2,

$$
\begin{aligned}
&(\mathcal{R}^s\zeta)(a_1,\dots,a_k) \\
= &\sum_{j=1}^{n}\sum_{s=1}^{k}\left(\nabla^2_{e_j,a_s}\zeta - \nabla^2_{a_s,e_j}\zeta\right)(e_j,a_1,\dots\widehat{a}_s\dots,a_k) + (\mathcal{T}^s\zeta)(a_1,\dots,a_k)\,.
\end{aligned}
\tag{5.7}
$$

Theorem 6.

$$
-\left(d^{s*}d^s\eta\right)(a_1,\dots,a_k) = \left(\operatorname{tr}\nabla^2\eta\right)(a_1,\dots,a_k) + \sum_{j=1}^{n}\sum_{s=1}^{k}\left(\nabla^2_{e_j,a_s}\eta\right)(e_j,a_1,\dots\widehat{a}_s\dots,a_k)
$$

for $\eta \in \mathscr{S}^k(A,E)$.

Proof. Let $\eta \in \mathscr{S}^k(A,E)$, $a_1,\dots,a_k \in \Gamma(A)$. Then

$$
\begin{aligned}
&-\left(\left(d^s\right)^* d^s\eta\right)(a_1,\dots,a_k) \\
= &\left(\operatorname{tr}\nabla d^s\eta\right)(a_1,\dots,a_k) \\
= &\sum_{j=1}^{n}\left(\nabla_{e_j}\left(d^s\eta\right)\right)(e_j,a_1,\dots,a_k) \\
= &\sum_{j=1}^{n}\nabla_{e_j}\left(\left(d^s\eta\right)(e_j,a_1,\dots,a_k)\right) - \sum_{j=1}^{n}\left(d^s\eta\right)\left(\nabla^A_{e_j}e_j,a_1,\dots,a_k\right) \\
&-\sum_{j=1}^{n}\sum_{s=1}^{k}\left(d^s\eta\right)\left(e_j,a_1,\dots,\nabla_{e_j}a_s,\dots,a_k\right) \\
= &\sum_{j=1}^{n}\nabla_{e_j}\left(\left(\nabla_{e_j}\eta\right)(a_1,\dots,a_k)\right) + \sum_{j=1}^{n}\sum_{s=1}^{k}\nabla_{e_j}\left(\left(\nabla_{a_s}\eta\right)(e_j,a_1,\dots\widehat{a}_s\dots,a_k)\right) \\
&-\sum_{j=1}^{n}\left(\nabla_{\nabla^A_{e_j}e_j}\eta\right)(a_1,\dots,a_k) - \sum_{j=1}^{n}\sum_{s=1}^{k}\left(\nabla_{a_s}\eta\right)\left(\nabla_{e_j}e_j,a_1,\dots\widehat{a}_s\dots,a_k\right) \\
&-\sum_{j=1}^{n}\sum_{s=1}^{k}\left(\nabla_{e_j}\eta\right)\left(a_1,\dots,\nabla^A_{e_j}a_s,\dots,a_k\right) \\
&-\sum_{j=1}^{n}\sum_{s=1}^{k}\left(\nabla_{\nabla^A_{e_j}a_s}\eta\right)(e_j,a_1,\dots\widehat{a}_s\dots,a_k) \\
&-\sum_{j=1}^{n}\sum_{s=1}^{k}\sum_{t\neq s}\left(\nabla_{a_t}\eta\right)\left(e_j,a_1,\dots,\nabla^A_{e_j}a_s,\dots\widehat{a}_t\dots,a_k\right).
\end{aligned}
$$

One can see that

$$\begin{aligned}
& \left(\operatorname{tr} \nabla^2 \eta\right)(a_1, \ldots, a_k) \\
= & \sum_{j=1}^{n} \left(\nabla^2_{e_j, e_j} \eta\right)(a_1, \ldots, a_k) \\
= & \sum_{j=1}^{n} \nabla_{e_j} \left(\left(\nabla_{e_j} \eta\right)(a_1, \ldots, a_k)\right) - \sum_{j=1}^{n} \sum_{s=1}^{k} \left(\nabla_{e_j} \eta\right)\left(a_1, \ldots, \nabla^A_{e_j} a_s, \ldots, a_k\right) \\
& - \sum_{j=1}^{n} \left(\nabla_{\nabla^A_{e_j} e_j} \eta\right)(a_1, \ldots, a_k)
\end{aligned}$$

and

$$\begin{aligned}
& \left(\nabla^2_{e_j, a_s} \eta\right)(e_j, a_1, \ldots \widehat{a}_s \ldots, a_k) \\
= & \nabla_{e_j} \left(\left(\nabla_{a_s} \eta\right)(e_j, a_1, \ldots \widehat{a}_s \ldots, a_k)\right) - \left(\nabla_{a_s} \eta\right)\left(\nabla^A_{e_j} e_j, a_1, \ldots \widehat{a}_s \ldots, a_k\right) \\
& - \sum_{t \neq s} \left(\nabla_{a_t} \eta\right)\left(e_j, a_1, \ldots, \nabla^A_{e_j} a_s, \ldots \widehat{a}_t \ldots, a_k\right) - \left(\nabla_{\nabla^A_{e_j} a_s} \eta\right)(e_j, a_1, \ldots \widehat{a}_s \ldots, a_k) .
\end{aligned}$$

Hence

$$\begin{aligned}
& -\left(\left(d^s\right)^* d^s \eta\right)(a_1, \ldots, a_k) \\
= & \left(\operatorname{tr} \nabla^2 \eta\right)(a_1, \ldots, a_k) + \sum_{j=1}^{n} \sum_{s=1}^{k} \left(\nabla^2_{e_j, a_s} \eta\right)(e_j, a_1, \ldots \widehat{a}_s \ldots, a_k) .
\end{aligned}$$

□

Theorem 7.

$$\left(d^s d^{s*} \eta\right)(a_1, \ldots, a_k) = \left(\mathcal{M}^s \eta\right)(a_1, \ldots, a_k) - \sum_{s=1}^{k} \sum_{j=1}^{n} \left(\nabla^2_{a_s, e_j} \eta\right)(e_j, a_1, \ldots \widehat{a}_s \ldots, a_k)$$

for $\eta \in \mathscr{S}^k(A, E)$.

Proof. Let $\eta \in \mathscr{S}^k(A, E)$, $a_1, \ldots, a_k \in \Gamma(A)$. Since

$$\begin{aligned}
& \left(\operatorname{tr} \nabla^2 \eta\right)(a_1, \ldots, a_k) \\
= & \sum_{j=1}^{n} \left(\nabla^2_{e_j, e_j} \eta\right)(a_1, \ldots, a_k) \\
= & \sum_{j=1}^{n} \nabla_{e_j} \left(\left(\nabla_{e_j} \eta\right)(a_1, \ldots, a_k)\right) - \sum_{j=1}^{n} \sum_{s=1}^{k} \left(\nabla_{e_j} \eta\right)\left(a_1, \ldots, \nabla^A_{e_j} a_s, \ldots, a_k\right) \\
& - \sum_{j=1}^{n} \left(\nabla_{\nabla^A_{e_j} e_j} \eta\right)(a_1, \ldots, a_k)
\end{aligned}$$

and

$$\begin{aligned}
&\left(\nabla^2_{a_s,e_j}\eta\right)(e_j,a_1,\dots\widehat{a}_s\dots,a_k)\\
=\ &\nabla_{a_s}\left(\nabla_{e_j}\eta\right)(e_j,a_1,\dots\widehat{a}_s\dots,a_k)-\left(\nabla_{\nabla^A_{a_s}e_j}\eta\right)(e_j,a_1,\dots\widehat{a}_s\dots,a_k)\\
=\ &\nabla_{a_s}\left(\left(\nabla_{e_j}\eta\right)(e_j,a_1,\dots\widehat{a}_s\dots,a_k)\right)-\left(\nabla_{e_j}\eta\right)\left(\nabla^A_{a_s}e_j,a_1,\dots\widehat{a}_s\dots,a_k\right)\\
&-\textstyle\sum_{t\neq s}\left(\nabla_{e_j}\eta\right)\left(e_j,a_1,\dots\widehat{a}_s\dots\nabla^A_{a_s}a_t\dots,a_k\right)-\left(\nabla_{\nabla^A_{a_s}e_j}\eta\right)(e_j,a_1,\dots\widehat{a}_s\dots,a_k),
\end{aligned}$$

by (5.4) and (5.6) we have

$$\begin{aligned}
&-\left((d^s)^*d^s\eta\right)(a_1,\dots,a_k)\\
=\ &\left(\operatorname{tr}\nabla d^s\eta\right)(a_1,\dots,a_k)\\
=\ &\sum_{j=1}^n\left(\nabla_{e_j}(d^s\eta)\right)(e_j,a_1,\dots,a_k)\\
=\ &\sum_{j=1}^n\nabla_{e_j}\left((d^s\eta)(e_j,a_1,\dots,a_k)\right)-\sum_{j=1}^n(d^s\eta)\left(\nabla^A_{e_j}e_j,a_1,\dots,a_k\right)\\
&-\sum_{j=1}^n\sum_{s=1}^k(d^s\eta)\left(e_j,a_1,\dots,\nabla^A_{e_j}a_s,\dots,a_k\right)\\
=\ &\sum_{j=1}^n\nabla_{e_j}\left(\left(\nabla_{e_j}\eta\right)(a_1,\dots,a_k)\right)+\sum_{j=1}^n\sum_{s=1}^k\nabla_{e_j}\left((\nabla_{a_s}\eta)(e_j,a_1,\dots\widehat{a}_s\dots,a_k)\right)\\
&-\sum_{j=1}^n\left(\nabla_{\nabla^A_{e_j}e_j}\eta\right)(a_1,\dots,a_k)-\sum_{j=1}^n\sum_{s=1}^k(\nabla_{a_s}\eta)\left(\nabla^A_{e_j}e_j,a_1,\dots\widehat{a}_s\dots,a_k\right)\\
&-\sum_{j=1}^n\sum_{s=1}^k\left(\nabla_{e_j}\eta\right)\left(a_1,\dots,\nabla^A_{e_j}a_s,\dots,a_k\right)-\sum_{j=1}^n\sum_{s=1}^k\left(\nabla_{\nabla^A_{e_j}a_s}\eta\right)(e_j,a_1,\dots\widehat{a}_s\dots,a_k)\\
&-\sum_{j=1}^n\sum_{s=1}^k\sum_{t\neq s}(\nabla_{a_t}\eta)\left(e_j,a_1,\dots,\nabla^A_{e_j}a_s,\dots\widehat{a}_t\dots,a_k\right)\\
=\ &\left(\operatorname{tr}\nabla^2\eta\right)(a_1,\dots,a_k)+\sum_{j=1}^n\sum_{s=1}^k\left(\nabla^2_{e_j,a_s}\eta\right)(e_j,a_1,\dots\widehat{a}_s\dots,a_k).
\end{aligned}$$

□

As a consequence of theorems 6, 7, definitions of $\mathcal{T}^s$, $\mathcal{M}^s$ and (5.7) we obtain the following formula on symmetric tensors.

Theorem 8. (*Weitzenböck-type Formula for Symmetric Forms*)

$$\Delta^s=\nabla^*\nabla-\mathcal{R}^s-\mathcal{M}^s+\mathcal{T}^s.$$

Notice that if ∇^A is a metric A-connection, then $\mathcal{M}^s=0$, and then $\Delta^s-\nabla^*\nabla=-\mathcal{R}^s+\mathcal{T}^s$. In the case where ∇^A is the Levi-Civita connection, the Weitzenböck formula for symmetric forms reduces to the shape:

$$\Delta^s=\nabla^*\nabla-\mathcal{R}^s.$$

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Formal Aspects of Topological Complexity

We study the concept of topological complexity from the viewpoint of fibrewise Lusternik-Schnirelmann category and discuss certain formal aspects which include the equivalence of various descriptions, the axiomatic characterization, and the possibility to obtain a decomposition into Δ*-sets of different dimensions.*

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1. Introduction

The concept of topological complexity was introduced by M. Farber in [4, 5] in his study of the navigation problem in robotics. Broadly speaking, the navigation problem refers to the problem of finding a continuous motion that transforms a mechanical system from some given initial position to a desired final position. To give a mathematical formulation of this problem one introduces the so-called *configuration space*, i.e. a topological space that describes all possible states of the mechanical system. For such a configuration space X one then considers the space X^I of all continuous paths $\alpha\colon I \to X$, and the evaluation map $\mathrm{ev}\colon X^I \to X \times X$ that to a path α assigns its end-points, $\mathrm{ev}(\alpha) := (\alpha(0), \alpha(1))$. A *navigation plan* for X is a rule that takes as input a pair of points $x, y \in X$, and returns as output a path α in X starting at x and ending at y. In other words, a navigation plan is a section of the evaluation map, i.e. a function $s\colon X \times X \to X^I$ such that $\mathrm{ev} \circ s = 1_{X\times X}$. Observe that while the movement through the configuration space is always assumed to be continuous with respect to the topology of the configuration space, this is not necessarily the case for the navigation plan. In fact, one can easily show that a continuous navigation plan exists if and only if X is contractible. Thus, for non-contractible spaces one is naturally led to consider navigation plans that are continuous only when restricted to subsets of $X \times X$.

Farber [4] exploited the fact that $\mathrm{ev}\colon X^I \to X \times X$ is a fibration, and defined the topological complexity of path-connected space X to be the Schwarz genus [19] of the fibration ev, i.e. the minimal n for which $X \times X$ can be covered by open subsets $U_1, \ldots, U_n$ such that each of them admits a continuous section $s_i\colon U_i \to X^I$ of ev. A very similar approach was previously used by S. Smale [20] and A. Vassiliev [21] in their investigation of the topological complexity of algorithms for finding roots of polynomial equations. Observe that strictly speaking, the sections $s_i\colon U_i \to X^I$ do not determine a navigation plan for X because the elements of the open cover of X must overlap, so over their intersections one has a multiple choice of navigation plans. To avoid this difficulty, one may decompose $X \times X$ into disjoint subsets such that the restriction of some global navigation plan to each of them is continuous. Clearly for a non-contractible configuration space, every such global navigation plan must be discontinuous, and that fact is sometimes described as the instability of the navigation planning algorithm. Farber [5] tackled this problem and proved that the topological complexity provides a suitable measure for the level of this instability.

It is clear from the definition that the topological complexity $\mathrm{TC}(X)$ is a homotopy invariant of X, and so it has recently attracted a lot of interest among homotopy theorists. This resulted in a series of interesting developments, variations and reformulations of the original idea. In particular, methods from the classical Lusternik-Schnirellman (LS) category, in particular the Whitehead-Ganea approach was developed in a series of papers [11], [12] and [13] by G. Calcines and L. Vandembroucq.

The alternative fibrewise LS category viewpoint was introduced by N. Iwase and M. Sakai in [15], and further applied and developed in [8], [9] and [10]. The fibrewise formulation avoids the use of function spaces, so the resulting theory has more geometric flavour and opens the possibility of extensive application of the methods of LS category to problems in topological complexity. In the first two section of this paper we use the Iwase-Sakai approach to give a uniform overview of known facts about the absolute and relative topological complexity together with slick and efficient proofs. The remaining sections exploit the alternative approach to obtain a couple of new results on the axiomatic approach to the topological complexity and on some useful dimension-wise decompositions.

2. Topological complexity as fibrewise category

In this section we show that the topological complexity of X can be described in terms of decompositions of the product $X \times X$ into subsets that can be deformed into the diagonal. and investigate the relations between different kinds of such decompositions.

Let X be a path-connected space and let $\mathrm{ev}\colon X^I \to X \times X$ be the evaluation fibration $\mathrm{ev}(\alpha) = \big(\alpha(0), \alpha(1)\big)$. A subset $F \subseteq X \times X$ *admits a continuous navigation plan* if there is a continuous map $s\colon F \to X^I$ such that $\mathrm{ev} \circ s = 1_F$. Various descriptions of the topological complexity of X are related to different ways to decompose of $X \times X$ into subsets that admit continuous navigation plans. We may broadly distinguish four different approaches as follows.

1. Originally [4] the topological complexity of X was defined as the Schwarz genus of the fibraton $\mathrm{ev}\colon X^I \to X \times X$. The Schwarz genus of a fibration $p\colon E \to B$ is the minimal n for which B can be covered by n open sets $U_1, \ldots, U_n$, such that each of them admits a continuous local section $s_i\colon U_i \to E$ of p. The use of open covers is standard in homotopy theory and allows direct comparison with other invariants. For example, recall that $\mathrm{cat}(X)$, the *Lusternik-Schnirelmann category* of X, is the minimal n for which X can be covered by n open sets $U_1, \ldots, U_n$, such that each $U_i \hookrightarrow X$ is null-homotopic, (i.e. each U_i can be deformed to a point inside X). One then have the following basic estimate (cf. [7, Section 4.2])

$$\mathrm{cat}(X) \leq \mathrm{TC}(X) \leq \mathrm{cat}(X \times X) \leq 2\,\mathrm{cat}(X) - 1 \,. \tag{2.1}$$

2. For applications in robotics the unavoidable overlapping of the sets of an open cover of $X \times X$ sometimes creates problems because it introduces a level of ambiguity on which navigation plan should be used for pairs of points that lie in the intersections. It is therefore often preferable to use partitions of $X \times X$ into disjoint subsets, so that the choice of the navigation plan is uniquely determined by the input data. Furthermore, we want to avoid subspaces with bad local properties. For that reason Farber [5] considered decompositions of $X \times X$ as disjoint unions of euclidean neighbourhood retracts. Recall that X is an *euclidean neighbourhood retract* (ENR) if it is homeomorphic to a retract of an open subset of some euclidean space $\mathbb{R}^n$. More intrinsically, X is an ENR if it is locally compact, locally contractible, and embeddable in some euclidean space (see [2, Section IV,8]). The class of ENR's contains all finite-dimensional cell complexes and all manifolds. Then one can consider global navigation plans for X that are continuous when restricted to the elements of some ENR-partition of $X \times X$ (i.e. a decomposition into a disjoint union of ENR's). For example, Farber [5] proved that for a connected polyhedron X the topological complexity of X equals the minimal n for which $X \times X$ has an ENR-partition into n subsets that admit continuous navigation plans.

3. Navigation plans that come up in applications are often defined locally, on small subsets of the product $X \times X$. For example, we can describe simple-minded navigation plans on a polyhedron X as follows. We first choose a maximal tree T in the 1-skeleton of X. Then for each pair of vertices $x, y \in X$ we define a navigation plan on the product of open stars $\mathrm{st}(x) \times \mathrm{st}(y)$ by combining the unique path in T between x and y with the straight segments in the respective stars. The number of elements in such a cover of $X \times X$ by sets admitting navigation plans is in general much bigger then $\mathrm{TC}(X)$. Since most of the elements are disjoint one may aggregate them to produce covers with less elements but this is usually impractical. There us however a different way to measure the complexity of such navigation plans. Given a cover $\mathcal{U}$ of X the *weight* of $\mathcal{U}$ is the maximal number of elements of $\mathcal{U}$ that have non-empty intersection. We will see later on that the weights if such covers are bounded bellow by the topological complexity of X.

4. Finally we can combine locally defined navigation plans with the requirement that their domains of definition are disjoint ENR's. Given a global navigation plan $s\colon X \times X \to X^I$ and some cover $\{F_\lambda\}$ of $X \times X$ by mutually disjoint ENR's, such that the restrictions $s|_{F_\lambda}$ are continuous, Farber [5] defined the *order of instability* of this partition to be the weight of the cover $\{\overline{F}_\lambda\}$. Once again, the topological complexity turns out to be the precise lower bound for the orders of instability of such partitions.

We now turn our attention from navigation plans to deformations of subsets of $X \times X$, starting from the following simple observation: every continuous navigation plan $s\colon F \to X^I$ by adjunction determines a homotopy $\widehat{s}\colon F \times I \to X \times X$, given by

$$\widehat{s}(x, y, t) := \big(x, s(x, y)(1-t)\big).$$

Since $s(x, y)(0) = x$ and $s(x, y)(1) = y$ the homotopy $\widehat{s}$ is clearly a vertical (i.e along the second factor) deformation of F to a subset of the diagonal $\Delta X = \{(x, x) \in X \times X\}$. This was already noted in [6, Section 18] and further developed by Iwase and Sakai in [15]. The main advantage of this alternative viewpoint is that a deformation of a space is much easier to visualize than a map into a path space.

Every subset of $X \times X$ that can be vertically deformed to a subset of the diagonal will be called Δ-*set*. Various characterizations of topological complexity are summarized in the following theorem.

Theorem 1. *If X is an ENR then the topological complexity of X equals the minimal n for which one (and hence all) of the following conditions is satisfied.*

(1) *There exists a cover of $X \times X$ by n open Δ-sets.*
(2) *There exists a cover of $X \times X$ by n closed Δ-sets.*
(3) *There exists an ENR-partition of $X \times X$ into n disjoint Δ-sets.*
(4) *There exists a filtration $\emptyset = F_0 \subseteq F_1 \subseteq \ldots \subseteq F_n = X \times X$ by closed subsets, such that each $F_i - F_{i-1}$ is a Δ-set.*
(5) *There exists a filtration $\emptyset = U_0 \subseteq U_1 \subseteq \ldots \subseteq U_n = X \times X$ by open subsets, such that each $U_i - U_{i-1}$ is a Δ-set.*
(6) *There exists a filtration $\emptyset = C_0 \subseteq C_1 \subseteq \ldots \subseteq C_n = X \times X$ by locally compact subsets, such that each $C_i - C_{i-1}$ is a Δ-set.*
(7) *$X \times X$ admits a cover of weight n by open Δ-sets.*
(8) *$X \times X$ admits a cover of weight n by closed Δ-sets.*
(9) *There exists an ENR-partition of $X \times X$ into disjoint Δ-sets, whose order of instability equals n.*

Proof. (1) is just a reformulation of the definition of the Schwarz genus. (2) is equivalent to (1) because as in the case of Lusternik-Schnirelmann category (cf. [17]) for spaces that are normal and neighbourhood retracts one can always work with closed instead of open coverings, and vice versa. (3) follows from [7, Proposition 4.9]. (4)-(6) correspond to the characterizations of [7, Proposition 4.12]. (7),(8) follow from [7, Corollary 4.14]. Finally (9) follows from [6, Theorem 13.1]. $\square$

We are now going to relate the characterization (1) in the above theorem to a special case of fibrewise Lusternik-Schnirelmann category. Take a Δ-set $U \subseteq X \times X$ and consider the projection $\pi \colon X \times X \to X$ of the product to the first factor. Restrictions of the homotopy that deforms U to the diagonal to the (possibly empty) intersections $V_x := U \cap \mathrm{pr}^{-1}(x) \subseteq \{x\} \times X$ yields a family of homotopies indexed by points of X that deform sets V_x within X to the point x. This precisely corresponds to the idea of a fibrewise deformation of set to a point, on which the following definition of fibrewise Lusternik-Schnirelmann category is based (cf. [18]). A *fibrewise pointed space* is a map $p \colon E \to B$ together with a section $s \colon B \to E$: we view this structure as a continuous family of pointed spaces $p^{-1}(b)$, each of them based at the point $s(b)$. Its *fibrewise Lusternik-Schnirelmann category* is the minimal n for which E can be covered by open sets $U_1, \ldots, U_n$ such that for each i there is a fibrewise homotopy deforming U_i to a subset of the section $s(B) \subset E$.

Let us consider the fibrewise pointed space over the base X whose total space is the product $X \times X$, $\pi \colon X \times X \to X$ is projection to the first factor and the section is given by the diagonal map $\Delta \colon X \to X \times X$. We will denote this fibrewise pointed space by $X \ltimes X$ where the semi-direct sign indicates that we have a 'twisted' familly of fibres indexed by the points of X, where the base 'acts' on the fibres by sliding the base-point. We may now conclude that $\mathrm{TC}(X)$ coincides with the fibrewise Lusternik-Schnirelmann category of $X \ltimes X$.

There are two important caveats regarding the role of the base-points (i.e. sections) that one must keep in mind when discussing the fibrewise category as related to the classical category. In the classical LS category the role of the base-points is minor, because for spaces with nice local behaviour the pointed and unpointed category coincide, and their value does not depend on the choice of the base-point. In fact, one can use the homotopy extension property and arrange that all sets of a categorical cover are deformed to the same point, and that all deformations are stationary at that point. Contrary to that, two sections of a fibrewise space may not be fibrewise homotopic, and the category with respect to one section can be completely different from the category with respect to some other section. For example the diagonal section of $\pi\colon S^2 \times S^2 \to S^2$ is clearly not homotopic to the constant section, and the fibrewise category of $\pi\colon S^2 \times S^2 \to S^2$ with respect to the diagonal section equals the topological complexity $\mathrm{TC}(S^2) = 3$, while the fibrewise category of π with respect to the constant section is the same as the ordinary category $\mathrm{cat}(S^2) = 2$.

The second point is even more delicate. First of all, we define (following [18]) the *fibrewise pointed category* of the fibrewise pointed space $p\colon E \to B$ with section $s\colon B \to E$ as the minimal n for which E can be covered by open sets $U_1, \ldots, U_n$ such that for each i $s(B) \subset U_i$ and the fibrewise homotopy deforming U_i to $s(B)$ is stationary on $s(B)$. The fibrewise pointed category is more adequate for the application of the homotopy-theoretical methods (cf. [18, Section 6], [15]), but it is not clear under what conditions the two notions coincide. In fact Iwase and Sakai [15] proposed a proof that pointed fibrewise category equals the unpointed fibrewise category for locally finite complexes but unfortunately their proof was flawed, see the Errata [16]. At the moment the best result in this direction is by A. Dranishnikov [3], who proved that the two versions of fibrewise category of X coincide when certain assumptions on the dimension of X are satisfied.

3. Subspace complexity

In this section we consider the topological complexity of subspaces of $X \times X$. We assume throughout that X is a Euclidean neighbourhood retract. Let $A \subseteq X \times X$ The *subspace topological complexity* of A, denoted $\mathrm{TC}_X(A)$ is the least integer n for which there exists a cover of A by n open Δ-subsets of $X \times X$. Of course, instead of covers by open sets we can use any of the equivalent descriptions of the topological complexity listed in Theorem 1. It is easy to see that the subspace complexity coincides with the relative complexity of A, which was defined in [7, Section 4.3] as the Schwarz genus of the restriction over A of the evaluation fibration $X^I \to X \times X$.

Let us list a few relations that follow immediately from the definition (most of them already appeared in the literature, cf. [7], Chapter 4 and in particular Section 4.3). First, we recover the topological complexity of X as

$$\mathrm{TC}(X) = \mathrm{TC}_X(X \times X). \tag{3.1}$$

If $X \subseteq Y$ and $A \subseteq B \subseteq X \times X$ then

$$\mathrm{TC}_Y(A) \leq \mathrm{TC}_X(B). \tag{3.2}$$

If $A, B \subseteq X \times X$ then

$$\mathrm{TC}_X(A \cup B) \leq \mathrm{TC}_X(A) + \mathrm{TC}_X(B). \tag{3.3}$$

Moreover, if A, B are separated open subsets of $X \times X$ (i.e. $\overline{A} \cap B = A \cap \overline{B} = \emptyset$) then

$$\mathrm{TC}_X(A \cup B) = \max\{\mathrm{TC}_X(A), \mathrm{TC}_X(B)\}. \tag{3.4}$$

The interplay between different characterizations given in Theorem 1 allows for unified and efficient proofs of the various estimates for topological complexity. To exemplify this approach we briefly summarize few most relevant results. we begin with a lemma that gives us plenty of Δ-sets.

Lemma 2. *Let X be a Euclidean neighbourhood retract.*

(1) *Any subspace of $X \times X$ that can be deformed within $X \times X$ into a Δ-set is itself a Δ-set. In particular, every product of two categorical subsets of X is a Δ-set (since it can be deformed to a point within $X \times X$).*
(2) *A union of a family of separated open Δ-sets is a Δ-set.*
(3) *If $h\colon X \ltimes X \to Y \ltimes Y$ is a homeomorphism of fibrewise pointed spaces then A is a Δ-set in $X \times X$ if, and only if $h(A)$ is a Δ-set in $Y \times Y$.*

Proof. (1) Let $A \subseteq X \times X$, and let $H\colon A \times I \to X \times X$ be a deformation of A, such that $A' = H_1(A)$ is a Δ-set. If we denote by $D'\colon A' \times I \to X \times X$ a vertical deformation of A' to the diagonal ΔX, than we obtain a vertical deformation D of A to the diagonal by the formula

$$\mathrm{pr}_2 \overline{D}(x, y, t) := \begin{cases} \mathrm{pr}_2\,(H(x, y, 3t)) & 0 \leq t \leq \frac{1}{3} \\ \mathrm{pr}_2\,(D'(H(x, y, 1), 3t - 1)) & \frac{1}{3} \leq t \leq \frac{2}{3} \\ \mathrm{pr}_1\,(H(x, y, 3 - 3t)) & \frac{2}{3} \leq t \leq 1 \end{cases}$$

(2) Recall that a family of subsets of a topological space is separated if the closure of each of them does not intersect the others. Clearly, when open Δ-sets are separated, then their deformations to the diagonal combine to a continuous deformation of their union to the diagonal.

3) A homeomorphism $h\colon X \times X \to Y \times Y$ is a homeomorphism of fibrewise pointed spaces if there is a homeomorphism $\bar{h}\colon X \to Y$ such that $\bar{h} \circ \pi_X = \pi_Y \circ h$ and $h \circ \Delta_X = \Delta_Y \circ \bar{h}$, so that the following diagram commutes

$$\begin{tikzcd} X \times X \arrow[r, "h"] \arrow[d, shift right, "\pi_X"'] & Y \times Y \arrow[d, shift right, "\pi_Y"'] \\ X \arrow[r, "\bar{h}"'] \arrow[u, shift right, "\Delta_X"'] & Y \arrow[u, shift right, "\Delta_Y"'] \end{tikzcd}$$

Then a deformation $H\colon A \times I \to X \times X$ of A to the diagonal Δ_X yields a deformation

$$\overline{H}\colon h(A) \times I \to Y \times Y, \qquad H(y, y', t) := h(H(h^{-1}(y, y'), t)$$

of $h(A)$ to the diagonal Δ_Y. $\square$

Part (1) of the above Lemma implies that every categorical subset of $X \times X$ is automatically a Δ-set, which immediately yields a relation between the subspace topological complexity and subspace category:

$$\mathrm{TC}_X(A) \leq \mathrm{cat}_{X\times X}(A). \tag{3.5}$$

If $B \subseteq X \times X$ can be deformed into some $A \subseteq X \times X$ (i.e., there is a deformation $H\colon B \times I \to X \times X$, such that $H_1(B) = H(B \times 1) \subseteq A$), then

$$\mathrm{TC}_X(B) \leq \mathrm{TC}_X(A). \tag{3.6}$$

In fact given a cover of A by Δ-sets $U_1, \ldots, U_n$, the pre-images $H_1^{-1}(U_1), \ldots, H_1^{-1}(U_n$ cover B and are also Δ-sets by (1) of Lemma 2. As a special case, if $B \subseteq X \times X$ can be deformed to its subset $A \subseteq B$, then by 3.2

$$\mathrm{TC}_X(A) = \mathrm{TC}_X(B). \tag{3.7}$$

Let X, Y be ENR's with $\mathrm{TC}(X) = m$ and $\mathrm{TC}(Y) = n$. Then by Theorem 1 (5) there exist a filtration $\emptyset = X_0 \subseteq X_1 \subseteq \ldots \subseteq X_m = X \times X$ such that all $X_i - X_{i-1}$ are Δ-sets in $X \times X$ and a filtration $\emptyset = Y_0 \subseteq Y_1 \subseteq \ldots \subseteq Y_n = Y \times Y$ such that all $Y_j - Y_{j-1}$ are Δ-sets in $Y \times Y$. If we define $Z_k := \bigcup_{i+j=k+1} X_i \times Y_j$ we obtain a filtration $\emptyset = Z_0 \subseteq Z_1 \subseteq \ldots \subseteq Z_{m+n-1} = (X \times Y) \times (X \times Y)$. We directly verify that $Z_k - Z_{k-1} = \coprod_{i+j=k+1}(X_i - X_{i-1}) \times (Y_j - Y_{j-1})$ is a disjoint union of separated Δ-sets, and conclude that

$$\mathrm{TC}(X \times Y) < \mathrm{TC}(X) + \mathrm{TC}(Y). \tag{3.8}$$

Let G be a topological group. If $U \subseteq G$ is an open categorical set, that can be deformed to the unit $e \in G$ then $\bigcup_{g\in G}\{g\} \times gU$ is clearly a Δ-set in $G \times G$. It follows that a categorical cover of G gives rise to a cover of $G \times G$ by Δ-sets, hence

$$\mathrm{TC}(G) = \mathrm{cat}(G). \tag{3.9}$$

4. Axiomatic characterization of topological complexity

Some of the properties listed in the previous section are sufficient to characterize precisely the subspace topological complexity among integer-valued functions with similar properties. In fact, we are going to show that the formulas 3.2, 3.3 and 3.6, together with a normalization requirement are sufficient to determine the topological complexity of a space. This approach is analogous to the axiomatic characterization of the Lusternik-Schnirelmann category as in [1].

Let us define the *abstract topological complexity* on a space X to be a function denoted $\mathrm{tc}(\cdot)$ that assigns a positive integer to every non-empty subset A of $X \times X$ and satisfies the following properties:

- **(tc1)** $\mathrm{tc}(\Delta X) = 1$;
- **(tc2)** If $A \subseteq B \subseteq X \times X$ then $\mathrm{tc}(A) \leq \mathrm{tc}(B)$;
- **(tc3)** If $A, B \subseteq X \times X$ then $\mathrm{tc}(A \cup B) \leq \mathrm{tc}(A) + \mathrm{tc}(B)$;
- **(tc4)** If $A, B \subseteq X \times X$, and B can be vertically deformed within $X \times X$ to a subset of A
  then $\mathrm{tc}(B) \leq \mathrm{tc}(A)$.

By the results from the previous section we know that the subspace topological complexity $\mathrm{TC}_X(\cdot)$ satisfies the conditions for the abstract topological complexity. We may now consider the set of all abstract topological complexities and order them as follows: if tc_1 and tc_2 are two abstract topological complexities, let

$$\mathrm{tc}_1(\cdot) \leq \mathrm{tc}_2(\cdot) \quad \Longleftrightarrow \quad \mathrm{tc}_1(A) \leq \mathrm{tc}_2(A) \text{ for all } A \subseteq X.$$

Let $\mathrm{tc}(\cdot)$ be an abstract topological complexity, and let U be a non-empty Δ-subset of $X \times X$. Then U can be vertically deformed to a subset of ΔX, so by (tc1) and (tc4) we have $\mathrm{tc}(U) \leq \mathrm{tc}(\Delta X) = 1$, therefore $\mathrm{tc}(U)$. Furthermore, If $A \subseteq X \times X$ can be covered by n open Δ-subsets $U_1, \dots, U_n$ of $X \times X$ then by (tc2) and (tc3)

$$\mathrm{tc}(A) \leq \mathrm{tc}(U_1 \cup \ldots \cup U_n) \leq \mathrm{tc}(U_1) + \ldots + \mathrm{tc}(U_n) = n.$$

Since $\mathrm{TC}_X(A)$ is precisely the minimal number of open Δ-subsets of $X \times X$ that are necessary to cover A we may conclude from the above discussion that

$$\mathrm{tc}(A) \leq \mathrm{TC}_X(A).$$

We have therefore proved the following result

Theorem 3. *The subspace topological complexity* $\mathrm{TC}_X(\cdot)$ *is the maximal element among all abstract topological complexities defined on subspaces of* $X \times X$.

5. Dimension-wise Δ-sets

The standard minimal decompositions of $S^n \times S^n$ into a disjoint union of ENR Δ-sets that yield the topological complexities of the spheres are well known. For odd-dimensional spheres we can take

$$A = \{(x, y) \in S^n \times S^n \mid x + y \neq 0\}$$

and

$$B = \{(x, y) \in S^n \times S^n \mid x + y = 0\},$$

and the dimensions are $\dim(A) = 2n$ and $\dim(B) = n$. On the other side, for even-dimensional spheres we may take

$$A = \{(x, y) \in S^n \times S^n \mid x + y \neq 0\},$$
$$B = \{(x, y) \in S^n \times S^n \mid x + y = 0\} - C,$$

and

$$C = \{(N, -N), (-N, N)\}$$

(where $N \in S^n$ denotes the north pole), and the respective dimensions of the sets involved are $2n, n$ and 0. One naturally wanders whether it is possible to achieve the same (i.e. Δ-sets of different dimensions) in the general case. We are going to prove this fact in the following form.

Theorem 4. *Let X be a connected ENR and let $A \subseteq X \times X$ be an ENR subset whose subspace topological complexity is* $\mathrm{TC}_X(A) = n$. *Then A can be decomposed as a disjoint union $A = X_1 \sqcup \ldots \sqcup X_n$, where each X_i is an ENR Δ-set and* $\dim(A) = \dim(X_1) > \dim(X_2) > \ldots \dim(X_n) \geq 0$.

In particular, if X is a connected ENR whose topological complexity is $\mathrm{TC}(X) = n$, *then $X \times X = X_1 \sqcup \ldots \sqcup X_n$, where X_i are ENR Δ-sets and* $2\dim(X) = \dim(X_1) > \dim(X_2) > \ldots \dim(X_n) \geq 0$.

The proof of the theorem is based on the following auxiliary result.

Lemma 5. *For every ENR subset $A \subseteq X \times X$ there exists an ENR subset $B \subset X \times X$ such that $\mathrm{TC}_X(A) > \mathrm{TC}_X(B)$, $\dim(A) > \dim(B)$ and $(A - B)$ is a Δ-set.*

Proof. For $TC_X(A) = 1$ we take $B := \emptyset$.

Let $\mathrm{TC}_X(A) = n$ and assume inductively that the claim holds for all $B \subseteq X \times X$ with $\mathrm{TC}_X(B) < n$. Let $U_1, \ldots, U_n$ be a cover of A by open Δ-sets in X. Then by the normality of X, and by the properties of the small inductive dimension, we can find an open set V_1 in X such that

$$A - U_2 - \ldots - U_n \subseteq V_1 \subseteq \overline{V}_1 \subseteq U_1,$$

and satisfying the requirement $\dim(\overline{V}_1 - V_1) < \dim(A)$. We can furthermore find an open cover $V_2, \ldots, V_n$ of $U_2 \cup \ldots \cup U_n$ such that $\overline{V}_i \subseteq U_i$ and $\dim(\overline{V}_i - V_i) < \dim(A)$.

Define $B := (\overline{V}_1 - V_1) \cup \ldots \cup (\overline{V}_n - V_n)$, so that clearly, $\dim(B) < \dim(A)$. Moreover, B is by the construction contained in the union $U_2 \cup \ldots \cup U_n$, hence $\mathrm{TC}_X(B) < \mathrm{TC}_X(A)$. Each component of $A - B$ is a Δ-set, as it contained in some U_i. Since the components of $A - B$ are separated 3.4 implies that $A - B$ itself is a Δ-set, which concludes the proof. $\square$

Proof. (of Theorem 4)
If $\mathrm{TC}_X(A) = n$ we can inductively apply the above lemma to obtain spaces $A = A_1 \supset A_2 \ldots \supset A_n \supset A_{n+1} = \emptyset$ such that $\dim(A_i) > \dim(A_{i+1})$ and $(A_i - A_{i+1})$ are ENR Δ-sets. To obtain the decomposition stated in the theorem we let $X_i := A_i - A_{i+1}$. Moreover, it is clear that $\dim(A) = \dim X_1$. $\square$

If X is a polyhedron with $\mathrm{TC}(X) = n$ then the above argument can be easily modified to obtain a filtration $\emptyset \leq X_1 \leq \ldots \leq X_n = X \times X$ by polyhedra whose dimension is strictly increasing, and such that each $X_i - X_{i-1}$ is a Δ-set. If X is $(p-1)$-connected then by Cellular approximation theorem every subcomplex of dimension less then p is a Δ-set, which implies that $\dim(X_2) \geq p$. It would be interesting to know (at least for the case when p divides $\dim(X)$) whether we can extend further the analogy with the spheres and obtain a filtration of $X \times X$ as above, by subpolyhedra whose dimensions are multiples of p.

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Fixed Points on Trivial Surface Bundles

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The main purpose of this work is to study fixed points of fiber- preserving maps over S^1 on the trivial surface bundles $S^1 \times S_2$, where S_2 is the closed orientable surface of genus 2. We classify all such maps that can be deformed fiberwise to a fixed point free map.

Introduction

Given a fibration $E \to B$ and $f : E \to E$ a fiber-preserving map over B, the question if f can be deformed over B (by a fiberwise homotopy) to a fixed point free map has been considered for several years by many authors. Among others, see for example [Dol74], [FH81], [Gon87], [Pen97], [GPV04], [GPV09I] and [GPV09II]. More recently also the fiberwise coincidence case has been considered in [Kos11], [GK09], [GPV10], [SV12], [Vie12] and [GKLN], which certainly has intersection with the fixed point case.

In [FH81], Fadell, E. and Husseini, S. showed that the fiberwise fixed point problem can be stated in terms of obstructions (including higher ones) if the fibration satisfies certain hypothesis. This is the case if the base space, the total space and the fiber F are manifolds, and the dimension of F is greater or equal to 3. The project to study fixed point of fiberwise maps for surface bundles has been considered mainly in the case where the base is S^1 and it can be divided into several cases as follows.

If the fiber F is the projective real space RP^2 we never obtain a fixed point free fiberwise map, because RP^2 has the fixed point property. This case leads to a natural question about the minimal size of the fixed point set, namely, when is possible to have the fixed point set connected. Close related, if not equivalent, is the problem of classify maps which can be deformed to a map with exactly one fixed point in each fiber.

The case of fiber S^2, despite the fact that the approach of [FH81] can be used, using different techniques, it was studied in [Kos11], [GPV10] and [GKLN].

We note that if the fiber is a closed surface S distinct of S^2 and RP^2, the approach of [FH81] can not be used. A project to study surface bundles for closed surface distinct of S^2 and RP^2 has started looking the case where the base is S^1. The case where S is the torus has been solved by other methods in [GPV04] (see also [Kos11]). For S the Klein bottle the results were obtained in [GPV09II] (see also [SV12]) by similar methods as the case of the torus.

In the present work we start the case of a surface bundle over S^1

where the fiber is S_2 and S_2 is the closed orientable surface of genus 2. More precisely we study fiberwise maps of the trivial bundle $S^1 \times S_2$.

Let us consider the fibration $S^1 \times S_2 \to S^1$ and $h : S^1 \times S_2 \longrightarrow S^1 \times S_2$ a fiber-preserving map over S^1, where $h(x, y) = (x, f(x, y)), \forall (x, y) \in S^1 \times S_2$ and f is a map from $S^1 \times S_2$ into S_2.

The main result of this paper is:

Theorem 4.3 A fiberwise map h can be deformed over S^1 to a fixed point free map if and only if h is fiberwise homotopic to $id \times g$ where $g : S_2 \to S_2$ is a fixed point free map homotopic to f restricted to $1 \times S_2$.

This paper is organized into 4 sections. In section 1 we review an approach to study fixed point of fiberwise maps and we adapt it for the case to be analyzed. In section 2 we make the main calculations where we compute the fundamental group of several spaces and homomorphisms to study a certain algebraic diagram. The main result of this section is Theorem 3.5. In section 3 we proof the main result of this work, which is Theorem 4.3. In section 4 we give a very brief view of the continuation of the study of the problems for the majority of the cases, which are still to be analyzed.

2 Preliminaries

Let $h : E \to E$ be a fiber-preserving map over B, i.e., $p \circ h = p$ where $p : E \to B$ is a fiber bundle with fiber a surface denoted by S. When is h deformable over B to a fixed point free map h' by a fiberwise homotopy over B ? We remark that in order to have a positive answer a necessary condition is that the map h restricted to a fiber is deformable to a fixed point free map.

Now we review an approach which was used in [GPV04] and [GPV09II]. Assuming the necessary condition, h is deformable over B to a fixed point free map h' by a fiberwise homotopy over B if and only if there exists a lifting ψ such that the following diagram is commutative, up to homotopy:

$$\begin{tikzcd}
 & \mathcal{F} \arrow[d] \\
 & \mathcal{E}(E \times_B E - \Delta) \arrow[d, "e_1"] \\
E \arrow[ur, dashed, "\psi"] \arrow[r, "(h,1)"'] & E \times_B E
\end{tikzcd} \tag{2.1}$$

Here $E \times_B E$ is the pullback of p by p, Δ is the diagonal in $E \times_B E$ and the inclusion $E \times_B E - \Delta \hookrightarrow E \times_B E$ is changed by the fibration $e_1 : \mathcal{E}(E \times_B E - \Delta) \to E \times_B E$ with fiber $\mathcal{F}$, where $\pi_i(\mathcal{F}) \simeq \pi_{i+1}(E \times_B E, E \times_B E - \Delta)$. Also $\mathcal{E}(E \times_B E - \Delta)$ is the pullback of the fibration $e_0 : (E \times_B E)^{[0,1]} \to E \times_B E$ by the inclusion $E \times_B E - \Delta \to E \times_B E$. The fibration $e_0 : (E \times_B E)^{[0,1]} \to E \times_B E$ is the evaluation at 0 and $e_1 : \mathcal{E}(E \times_B E - \Delta) \to E \times_B E$ is the evaluation at 1.

Let us observe that if E, B and S are closed manifolds then $\pi_{i+1}(E \times_B E, E \times_B E - \Delta) \simeq \pi_{i+1}(S, S - y_0)$ (see [FH81]).

When $E = B \times S$ is the trivial bundle and $h : B \times S \to B \times S$ is a fiber-preserving map over B, the map h can be write in the form $h(x, y) = (x, f(x, y))$ for some $f : B \times S \to S$. Then the diagram 2.1 can be modified and becomes equivalent to the following diagram:

$$
\begin{tikzcd}
 & \mathcal{F} \arrow[d] \\
 & \mathcal{E}(B \times (S \times S - \Delta)) \arrow[d, "e_1"] \\
B \times S \arrow[ur, dashed, "\psi"] \arrow[r, "(1,f,1)"'] & B \times S \times S
\end{tikzcd}
\tag{2.2}
$$

3 Trivial $S-$bundles over S^1 with $\chi(S) < 0$

Let S be a surface with $\chi(S) < 0$ and let us consider the fibration $S^1 \times S \to S^1$ and $h : S^1 \times S \longrightarrow S^1 \times S$ a fiber-preserving map over S^1, where $h(x, y) = (x, f(x, y))$, $\forall (x, y) \in S^1 \times S$ and f is a map from $S^1 \times S$ into S. We also consider x_0 and y_0 base points of S^1 and S, respectively, and $f : (S^1 \times S, (x_0, y_0)) \longrightarrow (S, f(x_0, y_0))$, with $f(x_0, y_0) \neq y_0$. From the map f we obtain the maps $g = f\mid_{\{x_0\}\times S}$ and $l = f\mid_{S^1\times\{y_0\}}$. Recall that we are assuming the necessary condition: *the map g is deformable to a fixed point free map.*

Using the approach developed in [GPV04] and [GPV09II] we will study in our case the existence of an algebraic lifting ψ to the diagram

$$
\begin{tikzcd}
 & 1 \arrow[d] \\
 & \pi_1(\mathcal{F}) \simeq \pi_2(S, S-x_0) \arrow[d] \\
 & \pi_1(\mathcal{E}(S^1\times(S\times S-\Delta))) \simeq \pi_1(S^1\times(S\times S-\Delta)) \arrow[d, "q_\#=1_{\pi_1(S^1)}\times j_\#"] \\
\pi_1(S^1\times S) \arrow[ur, dashed, "\psi"] \arrow[r, "(1,f,1)_\#"'] & \pi_1(S^1\times S\times S) \arrow[d] \\
 & 1
\end{tikzcd}
\tag{3.1}
$$

where $\pi_1(\mathcal{F}) \simeq \pi_1(S\times S-\triangle)$ is the pure braid group of S on 2-strings.

The existence of the lifting mentioned above is equivalent to find liftings θ and ϕ described in diagrams 3.2 and 3.3 below where θ and ϕ satisfy certain conditions. Since we are assuming the necessary condition then the lifting ϕ exists. So, we have the following two diagrams, where $i_{1\#}, i_{2\#}$ and $j_\#$ are induced homomorphisms on fundamental groups by the injective maps $i_1 : S^1 \to S^1\times S$, $i_2 : S \to S^1\times S$ and $j : S\times S-\Delta \to S\times S$, respectively, and $q_{2\#}$ and $p_{i\#}$ are induced homomorphisms by the projection maps $q_2 : S^1\times S\times S \to S\times S$ and $p_i : S\times S \to S$, respectively.

$$
\begin{tikzcd}
 & & \pi_1(S\times S-\Delta) \arrow[r, "p_i|_\#"] \arrow[dr, "j_\#"] & \pi_1(S) \\
\pi_1(S^1) \arrow[urr, dashed, "\theta"] \arrow[r, "i_{1\#}"'] & \pi_1(S^1\times S) \arrow[r, "(1,f,1)_\#"'] & \pi_1(S^1\times S\times S) \arrow[r, "q_{2\#}"'] & \pi_1(S\times S) \arrow[u, "p_{i\#}"]
\end{tikzcd}
\tag{3.2}
$$

$$\begin{tikzcd}
 & & \pi_1(S \times S - \Delta) \arrow[r, "p_{i|_\#}"] \arrow[dr, "j_\#"] & \pi_1(S) \\
\pi_1(S) \arrow[r, "i_{2\#}"'] \arrow[urr, "\phi"] & \pi_1(S^1 \times S) \arrow[r, "(1,f,1)_\#"'] & \pi_1(S^1 \times S \times S) \arrow[r, "q_{2\#}"'] & \pi_1(S \times S) \arrow[u, "p_{i\#}"]
\end{tikzcd} \tag{3.3}$$

We remark that in these diagrams we are omitting base points.

The following theorem provides some conditions that the liftings θ and ϕ must satisfy which are equivalent to a positive solution of the fixed point problem for the trivial bundle.

Theorem 3.1. *There exists ψ on the diagram 3.1 if and only if there exist θ and ϕ in the diagrams 3.2 and 3.3, respectively, such that Imθ commutes with Imϕ.*

Proof. Let us suppose that there exists a lifting ψ in the diagram (2.1). Define $\phi = q_{2|_\#} \circ \psi \circ i_{2\#}$ and $\theta = q_{2|_\#} \circ \psi \circ i_{1\#}$, where $i_1 : S^1 \to S^1 \times S$ and $i_2 : S \to S^1 \times S$ denote the inclusion maps and $q_{2|} : S^1 \times (S \times S - \Delta) \to (S \times S - \Delta)$ denotes the projection on the second factor. Therefore θ and ϕ are lifting for the diagrams 3.2 and 3.3, respectively, because $q_\# \circ \psi = (1, f, 1)_\#$ and $q_{2\#} \circ q_\# = j_\# \circ q_{2|_\#}$.

Now, for all $x \in \mathrm{Im}\theta$ and for all $y \in \mathrm{Im}\phi$ we have

$$\begin{aligned}
xy &= q_{2|_\#} \circ \psi \circ i_{1\#}([b]) q_{2|_\#} \circ \psi \circ i_{2\#}([s]) \\
&= q_{2|_\#} \circ \psi(i_{1\#}([b]) i_{2\#}([s])) \\
&= q_{2|_\#} \circ \psi(([b], 1)(1, [s])) \\
&= q_{2|_\#} \circ \psi((1, [s])([b], 1)) \\
&= q_{2|_\#} \circ \psi(i_{2\#}([s]) i_{1\#}([b])) \\
&= q_{2|_\#} \circ \psi \circ i_{2\#}([s]) q_{2|_\#} \circ \psi \circ i_{1\#}([b]) \\
&= yx
\end{aligned}$$

Conversely, suppose that θ and ϕ exist and we define ψ by $\psi([b], [s]) = ([b], \theta([b])\ \phi([s]))$ where $b : S^1 \to S^1$ and $s : S^1 \to S$ denote loops based at x_0 and at y_0, respectively. Since $\mathrm{Im}\theta$ commutes with $\mathrm{Im}\phi$, we have that ψ is a homomorphism and denoting by $s_0 : S^1 \to S$ and $b_0 : S^1 \to S^1$ the constant maps at y_0 and at x_0, respectively, it follows that $(q_{\#} \circ \psi)([b],[s]) = (1, f, 1)_{\#}([b],[s]), \forall([b],[s]) \in \pi_1(S^1 \times S)$. $\square$

From now on we specialize for the case where the fiber S is the surface S_2. So we consider the trivial bundle $S^1 \times S_2$.

If ϕ is a lifting of the diagram 3.3 to discuss the existence of the lifting θ we will denote by 1 a generator of $\pi_1(S^1) \equiv \mathbb{Z}$ and by $\theta(1) = \omega \in \pi_1(S_2 \times S_2 - \Delta)$. A presentation of $\pi_1(S_2 \times S_2 - \Delta)$ is given in [FH82]. We will use the following notation: let $a_i = \rho_{1,i} \in \pi_1(S_2 \times S_2 - \Delta), i = 1, 2, 3, 4$ and by $b_i = \rho_{2,i} \in \pi_1(S_2 \times S_2 - \Delta)$.

So $\pi_1(S_2 \times S_2 - \Delta)$ has the following presentation:

(I) $[a_1, a_2^{-1}][a_3, a_4^{-1}] =: B_1 = B_2^{-1} := [b_1, b_2^{-1}][b_3, b_4^{-1}]$ (which defines the elements B_1 and B_2^{-1}).

(II) $b_l a_j b_l^{-1} = a_j$ where $1 \leq j, l \leq 4$, and $j < l$(resp. $j < l - 1$) if l is odd (resp. l is even).

(III) $b_k a_k b_k^{-1} = a_k[a_k^{-1}, B_1]$ and $b_k^{-1} a_k b_k = a_k[B_1^{-1}, a_k]$ for all $1 \leq k \leq 4$.

(IV) $b_k a_{k+1} b_k^{-1} = B_1 a_{k+1}[a_k^{-1}, B_1]$ and $b_k^{-1} a_{k+1} b_k = B_1^{-1}[B_1, a_k]a_{k+1}[B_1^{-1}, a_k]$, for all k odd, $1 \leq k \leq 4$.

(V) $b_{k+1} a_k b_{k+1}^{-1} = a_k B_1^{-1}$, and $b_{k+1}^{-1} a_k b_{k+1} = a_k B_1 [B_1^{-1}, a_{k+1}]$, for all k odd, $1 \leq k \leq 4$.

(VI) $b_l a_j b_l^{-1} = [B_1, a_l^{-1}] a_j [a_l^{-1}, B_1]$ and $b_l^{-1} a_j b_l = [a_l, B_1^{-1}] a_j [B_1^{-1}, a_l]$ for all $1 \leq l < j \leq 4$ and $(j, l) \neq (2t, 2t - 1)$ for all $t \in \{1, 2\}$.

We also observe that from the fibration $p_2 \mid : S_2 \times S_2 - \Delta \longrightarrow S_2$ we get the following exact sequence:

$$1 \longrightarrow \pi_1(S_2 - y_0) \longrightarrow \pi_1(S_2 \times S_2 - \Delta) \xrightarrow{p_2|_{\#}} \pi_1(S_2) \longrightarrow 1 . \tag{3.4}$$

The group $\pi_1(S_2 - y_0)$ is free and from the sequence above it is identified with the subgroup of $\pi_1(S_2 \times S_2 - \Delta)$ freely generated by a_1, a_2, a_3, a_4. Also the image of the set of elements b_1, b_2, b_3, b_4 projects to a set of generators of $\pi_1(S_2)$ giving a presentation of $\pi_1(S_2) =< \bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4 \mid [\bar{b}_1, \bar{b}_2^{-1}][\bar{b}_3, \bar{b}_4^{-1}] >$. More details see [FH82].

Given a group G the central series of G is defined recursively by

$$G_1 = G, G_{n+1} = [G, G_n], n = 1, 2, \ldots.$$

For any group G we have that G_m is a normal subgroup of G_n for all $n \leq m$. In case G is free group of finite rank r then it is well known that G_n/G_{n+1} is a free abelian of rank

$$N_n = \frac{1}{n} \sum_{d|n} \mu(d) r^{\frac{n}{d}}$$

(see [[MKS76],Theorem 5.11, p.330]). Here $\mu(d)$ denotes the *Moebius Function* defined for all positive integers by $\mu(1) = 1$, $\mu(p) = -1$ if p is a prime number, $\mu(p^k) = 0$ for $k > 1$, and $\mu(b \cdot c) = \mu(b) \cdot \mu(c)$ if b and c are coprime integers.

For any group G denote the commutator $[[a, b], c]$ by (a, b, c). If a, b, c are elements of a group G and k, m, n are positive integers such that $a \in G_k, b \in G_m, c \in G_n$ then $(a, b, c) \cdot (b, c, a) \cdot (c, a, b) \equiv 1 \bmod G_{k+m+n+1}$ (see [[MKS76], Theorem 5.3, p.293]).

For the next Lemma, let $G = G_1 = \pi_1(S_2 - y_0)$ which is a free group, and it is identified with a subgroup of $\pi_1(S_2 \times S_2 - \Delta)$ using the short exact sequence 3.4.

Lemma 3.2. *If $v \in G_2 = [G_1, G_1]$, then $[b_j, v] \in G_3$, for $j = 1, 2, 3, 4$.*

Proof. We will prove the statement for b_3. The other cases are similar. If $v \in G_2 = [G_1, G_1]$, then v is a finite product of $[a_i, a_j]$ and of its inverses. If $v_1, v_2 \in G_2$ then $[b_3, v_1 v_2] = b_3 v_1 v_2 b_3^{-1} v_2^{-1} v_1^{-1} = b_3 v_1 b_3^{-1} v_1^{-1} v_1 b_3 v_2 b_3^{-1} v_2^{-1} v_1^{-1} = [b_3, v_1] v_1 [b_3, v_2] v_1^{-1}$. We know that the conjugation by v_1 preserves the central series and then if we prove that $[b_3,[a_i,a_j]]=1 \bmod G_3$ the result follows.

Now in G_1/G_3, we have that

$[b_3,[a_i,a_j]]=b_3[a_i,a_j]b_3^{-1}[a_i,a_j]^{-1}=b_3a_ia_ja_i^{-1}a_j^{-1}\ b_3^{-1}[a_i,a_j]^{-1}$.

In the case where $i\neq 4$ and $j\neq 4$ and recalling that in G_1/G_3, b_3 commutes with a_i and a_j (i.e, the action is trivial) we have the desired result.

If $i=4$ and $j\neq 4$ we also have that b_3 commutes with a_j because $j\neq 4$ and the action in a_4 results in B_1a_4. Therefore in G_1/G_3 the action of b_3 in $[a_4,a_j]$ is $b_3a_4a_ja_4^{-1}a_j^{-1}b_3^{-1}=B_1a_4a_ja_4^{-1}B_1^{-1}a_j^{-1}$ and so

$b_3[a_4,a_j]b_3^{-1}[a_4,a_j]^{-1}=B_1a_4a_ja_4^{-1}B_1^{-1}a_4a_j^{-1}a_4^{-1}=[B_1\,,\,a_4a_ja_4^{-1}]$
$=[a_4a_ja_4^{-1},B_1]^{-1}\in G_3$.

The case where $i\neq 4$ and $j=4$ is analogue. □

Let $C(\theta(1))$ be the centralizer of $\theta(1)$ in $\pi_1(S_2\times S_2-\Delta)$.

Proposition 3.3. Let θ be a lifting such that $(p_2\ |_{\#})(C(\theta(1))=\pi_1(S_2,y_0)$. Then $\theta(1)\in G$ and there exist u_1,u_2,u_3,u_4 elements of G such that $u_jb_j\in C(\theta(1))$, $j=1,2,3,4$. If $\theta(1)=xv$, with $v\in G_2, x\in G$, then we have that $[u_j^{-1},x^{-1}][x^{-1},b_j]=0$ in G_2/G_3.

Proof. From the hypothesis $(p_2\ |_{\#})(C(\theta(1)))=\pi_1(S_2,y_0)$ follows that $p_{2\#}(\theta(1))$ is in the centralizer of $\pi_1(S)$. This implies that this element is trivial and then $\theta(1)\in G$. Also, given $\bar b_j$ from the hypothesis follows that there exists x_j which is in the centralizer of $\theta(1)$ which projects to $\bar b_j$. Therefore $x_j=u_jb_j$ for some $u_j\in G$.

Since $u_jb_j\in C(\theta(1))$, $j=1,2,3,4$ we have that

$$\begin{aligned} xvu_jb_j &= u_jb_jxv \\ b_jxb_j^{-1}[b_j,v] &= u_j^{-1}xu_j[u_j^{-1},v] \\ u_j^{-1}x^{-1}u_jb_jxb_j^{-1} &= [u_j^{-1},v][v,b_j] \\ [u_j^{-1},x^{-1}][x^{-1},b_j] &= [u_j^{-1},v][v,b_j] \end{aligned}$$

where $v \in G_2$, $u_j \in G_1$. Then in G_2/G_3 it follows from the Lemma 3.2 that

$$[u_j^{-1}, x^{-1}][x^{-1}, b_j] = 0. \tag{3.5}$$

□

The group G_2/G_3 is a **Z-** free module and let us consider the basis

$$\{[a_1, a_2], [a_1, a_3], [a_1, a_4], [a_2, a_3], [a_2, a_4], [a_3, a_4]\}$$

which we refer as the canonical basis of G_2/G_3.

Lemma 3.4. *In G_2/G_3 we have:*

a) $[a_i a_j, x] = [a_j a_i, x]$, *where a_i, a_j are generators of G and $x \in G$.*

b) If $B = [a_1, a_2^{-1}][a_3, a_4^{-1}] \in G$, then $B = -[a_1, a_2] - [a_3, a_4]$ and its coordinate in relation to the canonical basis is given by $(-1, 0, 0, 0, 0, -1)$.

c) The element $[a_i^{x_i}, a_j^{x_j}]$ is given in the following form:

$$[a_i^{x_i}, a_j^{x_j}] = \begin{cases} 0 \ se \ i = j \\ x_i x_j [a_i, a_j] \ se \ i < j \\ -x_i x_j [a_j, a_i] \ se \ i > j \end{cases}$$

Proof. Since $[a_i a_j, x] = [a_i, [a_j, x]][a_j, x][a_i, x]$ and $[a_j a_i, x] = [a_j, [a_i, x]][a_i, x][a_j, x]$, the result of item a) follows by observing that in G_2/G_3 we have that $[a_i, [a_j, x]] = 0 = [a_j, [a_i, x]]$ which is commutative. The items b) and c) are easy. □

Theorem 3.5. *If there exists θ and $(p_2 \mid_{\#})(C(\theta(1)) = \pi_1(S_2, y_0)$ then $\theta(1) \in G_2 = [G_1, G_1]$.*

Proof. It follows from the exact sequence $1 \to [G_1, G_1] \to G_1 \to G_1/[G_1, G_1] \to 0$ that $\theta(1) \in G_1$ is of the form $\theta(1) = xv$ with

$x = a_1^a a_2^b a_3^c a_4^d$ and $v \in G_2 = [G_1, G_1]$. We are going to prove that the exponents $a = b = c = d = 0$.

It follows from Proposition 3.3 that in G_2/G_3 we have $[u_j^{-1}, x^{-1}][x^{-1}, b_j] = 0$ with $u_j \in G_1$.

In fact this is a system with j equations and four variables x:(a, b, c, d) and for each j four variables (e_j, f_j, g_j, h_j) corresponding to $u_j^{-1} = a_1^{e_j} a_2^{f_j} a_3^{g_j} a_4^{h_j}$.

Writing $[u^{-1}, x^{-1}]$ in the canonical basis of G_2/G_3 and observing that the exponents of $u^{-1} = a_1^e a_2^f a_3^g a_4^h$ must to appear with sub-index j (We are omitting such sub-index) we obtain:

$[u^{-1}, x^{-1}] = (af - be)[a_1, a_2] + (ag - ce)[a_1, a_3] + (ah - de)[a_1, a_4] + (bg - cf)[a_2, a_3] + (bh - df)[a_2, a_4] + (ch - dg)[a_3, a_4]$

Calculating $[x^{-1}, b_j] \in G_2$ with $x = a_1^a a_2^b a_3^c a_4^d$ we obtain:

$$\begin{aligned}
x^{-1}b_1xb_1^{-1} &= x^{-1}a_1^a(B_1a_2)^b a_3^c a_4^d \\
&= a_4^{-d}a_3^{-c}a_2^{-b}(B_1a_2)^b a_3^c a_4^d \\
&= a_4^{-d}a_3^{-c}a_2^{-b}(a_2^b a_2^{-b}B_1a_2^b \ldots a_2^{-2}B_1a_2^2a_2^{-1}B_1a_2)a_3^c a_4^d \\
x^{-1}b_2xb_2^{-1} &= x^{-1}(a_1B_1^{-1})^a a_2^b a_3^c a_4^d \\
&= a_4^{-d}a_3^{-c}a_2^{-b}a_1^{-a}(a_1B_1^{-1})^a a_2^b a_3^c a_4^d \\
&= a_4^{-d}a_3^{-c}a_2^{-b}a_1^{-a}(a_1B_1^{-1}a_1^{-1}a_1^2B_1^{-1}a_1^{-2} \ldots a_1^aB_1^{-1}a_1^{-a})\times \\
&\quad \times a_1^a a_2^b a_3^c a_4^d \\
x^{-1}b_3xb_3^{-1} &= x^{-1}a_1^a a_2^b a_3^c(B_1a_4)^d \\
&= a_4^{-d}(B_1a_4)^d \\
&= a_4^{-d}(a_4^d a_4^{-d}B_1a_4^d \ldots a_4^{-2}B_1a_4^2a_4^{-1}B_1a_4) \\
x^{-1}b_4xb_4^{-1} &= x^{-1}a_1^a a_2^b(a_3B_1^{-1})^c a_4^d \\
&= a_4^{-d}a_3^{-c}(a_3B_1^{-1})^c a_4^d \\
&= a_4^{-d}a_3^{-c}(a_3B_1^{-1}a_3^{-1}a_3^2B_1^{-1}a_3^{-2} \ldots a_3^cB_1^{-1}a_3^{-c}a_3^c)a_4^d
\end{aligned}$$

Therefore in relation to the canonical basis of G_2/G_3 and by using Lemma 3.4 *b*) we obtain $(-b, 0, 0, 0, 0, -b), (a, 0, 0, 0, 0, a), (-d, 0, 0, 0, 0, -d), (c, 0, 0, 0, 0, c)$ as the coordinates of $[x^{-1}, b_j]$ respectively for $j = 1, 2, 3, 4$.

So, the system to be solved is

$$\begin{cases} -be & +af & & & = & b, -a, d, -c \text{ respectively for } b_1, b_2, b_3, b_4 \\ -ce & & +ag & & = & 0 \\ -de & & & +ah & = & 0 \\ & -cf & +bg & & = & 0 \\ & -df & & +bh & = & 0 \\ & & -dg & +ch & = & b, -a, d, -c \text{ respectively for } b_1, b_2, b_3, b_4 \end{cases} \tag{3.6}$$

understanding that in the letters (e, f, g, h) must to appear sub-index j, but not in the letters (a, b, c, d).

a-) $d \neq 0$ in the system (3.6)

a1) If $b = 0$ we have that L_5 implies $f = 0$, making L_1 without solution.

a2) If $b \neq 0$, in the system $L_3 \to dL_1 - bL_3$ produces an incompatibility : new L_3 and L_5 .

b-) $b \neq 0$ in the system (3.6)

b1) If $d = 0$ we have that L_6 implies $c \neq 0$ and $h \neq 0$. Then L_3 implies $a = 0$ and from L_5 we conclude that $b = 0$, making L_1 without solution.

b2) If $d \neq 0$, in the system $L_4 \to bL_6 + dL_4$ produces an incompatibility: new L_4 and L_5.

c-) $c \neq 0$ in the system (3.6)

c1) If $a = 0$ then L_1 implies that $b \neq 0$ and $e \neq 0$ and from L_3 we obtain $d = 0$ and from L_5 we have that $h = 0$. Therefore L_6 is impossible.

c2) If $a \neq 0$ the system is impossible. In the system we make $L_4 \to cL_1 - aL_4$ and obtain an incompatibility: new L_4 and L_2.

d-) $a \neq 0$ in the system (3.6)

d1) If $c = 0$ the system is impossible. It follows from the fact that L_6 implies $d \neq 0$ and $g \neq 0$. Also L_4 implies $b = 0$ and L_5 implies $f = 0$ making L_1 impossible.

d2) If $c \neq 0$ the system is impossible, because in the system we make $L_3 \to aL_6 - cL_3$ which produces an incompatibility: new L_3 and L_2.

From the considerations above we conclude that $a = b = c = d = 0$ and therefore $x = 1$ and $\theta(1) \in G_2$. $\square$

4 Main Result

Let $h : S^1 \times S_2 \to S^1 \times S_2$ given by $h(x, y) = (x, f(x, y))$. Let us consider $l : (S^1, x_0) \to (S_2, f(x_0, y_0))$ and $g : (S_2, y_0) \to (S_2, f(x_0, y_0))$ given by $l(x) = f(x, y_0)$ and $g(y) = f(x_0, y)$, respectively. Without loss of generality we are assuming that g is a fixed point free map.

To prove our main result we need the following

Lemma 4.1. *Let $t : S_2 \to S_2$ be a continuous map and $t_\# : \pi_1(S_2) \to \pi_1(S_2)$ the induced homomorphism of the map t. Suppose that $t_\#(b_i) = \alpha^{n_i}$, where $b_i, i = 1, 2, 3, 4$ is a generator of $\pi_1(S_2)$. If the map t can be deformed to a fixed point free map then $\sum_1^4 n_i |\alpha|_i = 1$ where $|\alpha|_i$ denotes the sum of the exponents of b_i in the word α.*

Proof. Let $\iota : S^1 \to S_2$ be a map which represents the element $\alpha \in \pi_1(S_2)$. We can define $t' : S_2 \to S^1$ such that $\iota \circ t' = t$. By the commutativity property for fixed point we know that the Nielsen number of t is the same as the Nielsen number of $t' \circ \iota$, which is a self map of the circle. So if t is deformable to a fixed point free map then we have that the Nielsen number of $t' \circ \iota$ is trivial which is equivalent to say that the

Lefschetz number of $t' \circ \iota$ is 0, which is the same to say $\sum_1^4 n_i|\alpha|_i = 1$. So the result follows. □

The main result will follows from the Proposition below.

Proposition 4.2. The fiberwise map h is deformable to a fixed point free map over S^1 if and only if $l_\#(1) = e$, where $l_\# : \pi_1(S^1; x_0) \to \pi_1(S_2; f(x_0, y_0))$.

Proof. Let h be a fiberwise map where $h(x, y) = (x, f(x, y))$. To prove that h can be deformed fiberwise to a fixed point free map it is suffice to show that f is homotopic to the map $f^{'}(x, y) = f(x_0, y)$.

Because $S^1 \times S_2$ and S_2 are $K(\pi, 1)$ the two maps are homotopic if the induced homomorphisms on the fundamental group are equal. Because $\pi_1(S^1 \times S_2) = \pi_1(S^1) \times \pi_1(S_2)$ to show that the two homomorphisms are the same it suffices to show that these homomorphisms coincide when restricted to each of the two subgroups $\pi_1(S^1), \pi_1(S_2)$. By hypothesis $l_\#(1) = e$ follows that they coincide on $\pi_1(S^1)$. By the definition of $f^{'}$ also follows that they coincide on $\pi_1(S_2)$, and this concludes the proof of one implication.

Reciprocally, let h be a map deformable to a fixed point free map over S^1. Then by Theorem 3.1 exist ϕ and θ such that the image of θ commutes with the image of ϕ. From the diagrams 3.2 and 3.3, ${p_1}_{|\#} \circ \theta = l_\#$ and ${p_1}_{|\#} \circ \phi = g_\#$. It is known that $g_\#(\pi_1(S_2))$ is a subgroup of $\pi_1(S_2)$ isomorphic to one of the following groups:

1. $\{e\}$.
2. a free group of rank 2 (see [LS89]and [Zie62]).
3. $\pi_1(S)$
4. $\mathbb{Z} = \langle \beta \rangle$

The first item does not occur, otherwise g is homotopic to the constant map so it can not be deformed to a fixed point free map.

In the second and third cases, from above $l_{\#}(1)$ commutes with all elements of $g_{\#}(\pi_1(S))$ but the centralizer of these two subgroups is trivial. Therefore $l_{\#}(1) = e$.

For the last case we have that $g_{\#}(\pi_1(S)) = \mathbb{Z} = \langle \beta \rangle = \langle \alpha^k \rangle$ where $\alpha \neq 0$, α has no roots and $\alpha^k = \beta$. Since $l_{\#}(1)$ commutes with the elements of $g_{\#}(\pi_1(S))$ then $l_{\#}(1) = \alpha^r$. If $r = 0$ the proof follows. So suppose that $r \neq 0$.

Writing $g_{\#}(b_i) = \alpha^{n_i}$ we have by the lemma 4.1 that if g is homotopic to a fixed point free map then $\sum_1^4 n_i |\alpha|_i \neq 0$.

We have that $p_{1|_{\#}} \circ \theta(1) = l_{\#}(1)$. We also have that $\text{im}\ \phi \subset C(\theta(1))$, and $p_{2|_{\#}}(\text{Im}(\phi))) = \pi_1(S)$. Therefore $p_{2|_{\#}}(C(\theta(1))) = \pi_1(S)$ and by theorem 3.5 follows that $\theta(1) \in G_2 = [G_1, G_1]$.

So, $\alpha^r = l_{\#}(1) \in \left[p_{1|_{\#}}(G_1), p_{1|_{\#}}(G_1)\right]$.

Therefore $\alpha \in \left[p_{1|_{\#}}(G_1), p_{1|_{\#}}(G_1)\right]$ and then $|\alpha|_i = 0$ and by using the above result we conclude that g is not homotopic to a fixed point free map, which contradicts the initial condition on g. So the result follows. $\square$

In fact the proof above shows that if $\theta(1) = e$ then f does not depend of x, i.e. h is the unique fiberwise map homotopic to $id \times g$ where $g : S_2 \to S_2$ is a fixed point free map homotopic to f restricted to $1 \times S$. So we state the main result.

Theorem 4.3. *A fiberwise map h can be deformed over S^1 to a fixed point free map if and only if h is fiberwise homotopic to $id \times g$ where $g : S_2 \to S_2$ is a fixed point free map homotopic to f restricted to $1 \times S_2$.*

5 Other surface bundles

Here let us make few comments about the fixed point question studied in the previous sections in the case we have a more general surface bundle. Let $S \to E \to B$ be a surface bundle over a space B where S is a closed surface of negative Euler characteristic. We can consider three subfamilies of the family of these bundles, namely: I) let S be an arbitrary closed surface(orientable or nonorientable) of arbitrary genus $g > 1$, and $E = S^1 \times S$; II) let $E = B \times S$ be a bundle for B any connected CW complex; III) let E be a $S-$bundle over S^1.

The subcases I) and II) we expect that the answer of the problem should be similar to the answer of the case studied here where $S = S_2$.

The subcase III) is more subtle. First of all the formulation of the problem is already more elaborate. More precisely, let us consider the map $\phi : [E, E]_B \to [S, S]$ which associate to a homotopy class of a fibre preserve map $[f]$ the homotopy class of the restriction $f|_S : S \to S$. Then one would like to know first which homotopy class $[g] \in [S, S]$ which contains a fixed point free map are in the image of ϕ. Second, for a class $[g]$ in the image how many classes $[f] \in [E, E]$ we would like to compute the pre-image of $[g]$, i.e. $\phi^{-1}[g]$. For example in the case that we solved, we have that the $[g]$ is in the image for all maps g which are fixed point free and the pre-image contains exactly one element.

The study and full calculation of the questions above are in progress and should appear somewhere.

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Free $\mathcal{A}_4$-actions on products of spheres

We summarize what is known about free actions of $\mathcal{A}_4$, the alternating group on four letters, on products of spheres. New results are also included: in particular, we prove that $\mathcal{A}_4$ acts freely on $S_n \times S_n \times S_n$ if and only if $n = 1, 3, 7$.

1 Introduction

The following result first appeared in a 1979 paper of Oliver.

Proposition 1.1 ([15, p. 547]). *Let G be a finite group. For any integer $n \geq 1$, there exists an integer $k \geq 1$ such that G acts freely on $(S^{2n-1})^k$.*

Proof. Let G be a finite group, $H \subseteq G$ a subgroup and X an H-space. Then the space $\mathrm{Map}_H(G, X)$ of all H-equivariant maps $G \to X$, endowed with the compact-open topology, is a G-space in the obvious way. Note that if X is a free H-space, then $\mathrm{Map}_H(G, X)$ is also a free H-space. Furthermore,

$$\mathrm{Map}_H(G, X) \approx X^{[G\,:\,H]},$$

where $[G : H]$ denotes the index of H in G; the homeomorphism is given by the evaluation on a set of representatives of cosets of H.

For any non-trivial element $g \in G$, the cyclic group $\langle g \rangle \subseteq G$ acts freely on any odd-dimensional sphere S^{2n-1}. Then G acts on

$$M_g = \mathrm{Map}_{\langle g \rangle}(G, S^{2n-1}) \approx (S^{2n-1})^{[G\,:\,\langle g \rangle]},$$

with the subgroup $\langle g \rangle \subseteq G$ acting freely. The product $\prod_{g \in G,\, g \neq 1} M_g$ with the diagonal G-action is a free G-space. $\square$

The downside of the construction outlined in Proposition 1.1 is that it is very inefficient: the number of spheres in the resulting product is *very* unlikely to be minimal. For example, for G the elementary abelian p-group of rank r, it yields an action on $(S^{2n-1})^{p^{r-1}(p^r-1)}$, while such a group clearly acts freely on $(S^{2n-1})^r$. This raises an interesting problem:

Given a finite group G, determine the minimal number $k = k(G)$ such that G acts freely on a finite CW complex homotopy equivalent to $(S^n)^k$ for some $n \geq 1$.

A lot of effort has been put into determining k for various classes of groups. For example, the solution of the spherical space form problem asserts that $k(G) = 1$ if and only if G has periodic cohomology (see [9]).

In [5], we investigated k for the class of simple alternating groups, i.e., with $\mathcal{A}_4$ excluded; the main result is that $k(\mathcal{A}_d) > d - 1$ for many values of $d \geq 5$. Our goal for this article is twofolds. Firstly, we want to summarize what is known about and completely understand $k(\mathcal{A}_4)$, building on previous insights provided by Oliver [15] and Plakhta [16]. This is achieved in Section 3, and the main results there are that $\mathcal{A}_4$ cannot act freely on any finite-dimensional CW complex homotopy equivalent to $S^n \times S^n$ (Proposition 3.2), and that $\mathcal{A}_4$ acts freely on $S^n \times S^n \times S^n$ if and only if $n = 1, 3, 7$ (Theorem 3.4). From this point of view, this article can be seen as complementary to [5].

Secondly, in Section 4, we explain how $\mathcal{A}_4$ can act freely on $S^m \times S^n$ if $m \neq n$. Then we look at those pairs (m, n) for which there exists a free $\mathcal{A}_4$-action on $S^m \times S^n$.

Apart from that, Section 5 is dedicated to free actions of the symmetric group on three letters $\mathcal{S}_3$. This is intended mainly for the sake of completeness, but the methods presented therein also generalize the the class of dihedral groups.

Notation. All considered actions are topological, i.e., by homeomorphisms. A 'closed manifold' is taken to mean a compact and connected manifold without boundary.

2 Preliminaries

Results of Subsections 2.1 and 2.2 are indispensable to the whole Section 3. Subsection 2.3 is relevant only to Example 3.5.

2.1 Integral representations of $\mathbb{Z}_3$

Write $GL(n, \mathbb{Z})$ for the general linear group of degree n over the integers.

Lemma 2.1. (1) *Up to conjugation, there exists precisely one subgroup of order 3 in $GL(2, \mathbb{Z})$:*

$$\left\langle \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \right\rangle.$$

(2) *Up to conjugation, there exist two subgroups of order 3 in $GL(3, \mathbb{Z})$:*

$$\left\langle \begin{bmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\rangle, \left\langle \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right\rangle.$$

Lemma 2.1 is a consequence of the general theory of integral representations of cyclic groups of prime order (see [8, §74]), but can also be derived by elementary calculations.

2.2 Adem's results

Recall that if a group G acts on a space X, then the cohomology groups of X assume the structure of a G-module. We will need the following result due to Adem, which relates the nature of the G-action on X and the G-module structure on $H^n(X; \mathbb{Z})$ in the case when G is a cyclic group of prime order and X is a product of n-dimensional spheres.

Theorem 2.2 ([1, Corollary 4.8])**.** *Let k, n be positive integers, p an odd prime. If $\mathbb{Z}_p$ acts freely on a finite-dimensional CW complex X such that the cohomology rings $H^*(X;\mathbb{Z})$ and $H^*\big((S^n)^k;\mathbb{Z}\big)$ are isomorphic, then $H^n(X;\mathbb{Z})$ splits off a trivial direct summand as a $\mathbb{Z}_p$-module.*

We will also make use of the following basic observation, again due to Adem.

Proposition 2.3 ([1, Proposition 2.1])**.** *Let $n \neq 1, 3, 7$. If $f\colon (S^n)^k \to (S^n)^k$ is a map such that $f^*\colon H^n\big((S^n)^k;\mathbb{Z}\big) \to H^n\big((S^n)^k;\mathbb{Z}\big)$ is an automorphism, then the modulo 2 reduction of f^* is a permutation matrix in the usual basis.*

2.3 Borel manifolds

Recall that a closed manifold M is *aspherical* if the higher homotopy groups of M vanish, i.e., if $\pi_i(M) = 0$ for $i \geq 2$, or, equivalently, if the universal cover of M is contractible. It is classically known that aspherical manifolds are classified up to homotopy by their fundamental groups: two aspherical manifolds are homotopy equivalent if and only if they have isomorphic fundamental groups.

On the geometric level, we have Borel manifolds: a closed manifold M is called a *Borel manifold* if every closed manifold homotopy equivalent to M is automatically homeomorphic to M. Crucially for us, examples of Borel manifolds include compact solvmanifolds (see [4, Chapter III, Section 4]); in particular, products of circles.

Proposition 2.4. *Let M be an n-dimensional aspherical Borel manifold. A finite group G acts freely on M if and only if there exists a group extension*

$$1 \to \pi_1(M) \to \pi_1(N) \to G \to 1,$$

where N is a closed n-dimensional aspherical manifold.

Proof. Suppose a finite group G acts freely on M. The orbit space M/G is well-known to be a closed n-dimensional manifold. Inspection of the long exact sequence of homotopy groups of the covering $M \to M/G$ reveals that M/G is aspherical and that $\pi_1(M/G)$ fits into the extension $1 \to \pi_1(M) \to \pi_1(M/G) \to G \to 1$.

Conversely, consider a group extension

$$1 \to \pi_1(M) \to \pi_1(N) \to G \to 1.$$

Let $\tilde{M}$ be the covering space of N corresponding to the subgroup $\pi_1(M) \subseteq \pi_1(N)$. Then $G \cong \pi_1(N)/\pi_1(M)$ acts freely on $\tilde{M}$. Since $\tilde{M}$ is a closed aspherical manifold with $\pi_1(\tilde{M}) \cong \pi_1(M)$, we have that $\tilde{M}$ is homotopy equivalent to M. But M is a Borel manifold by hypothesis, so $\tilde{M}$ and M are homeomorphic, and the conclusion follows. □

Remark 2.5. The famous Borel conjecture states that every closed, aspherical manifold is a Borel manifold. (See [10] for more details.)

3 Free $\mathcal{A}_4$-actions on $(S^n)^k$

Recall the following fundamental result due to Oliver:

Theorem 3.1 ([15, Theorem 1]). *Let k, n be positive integers. If the alternating group $\mathcal{A}_4$ acts freely on a finite-dimensional CW complex X such that the cohomology rings $H^*(X;\mathbb{Z}_2)$ and $H^*\big((S^n)^k;\mathbb{Z}_2\big)$ are isomorphic, then the action induced on $H^n(X;\mathbb{Z}_2)$ is non-trivial.*

Oliver combined Theorem 3.1 and the Lefschetz Fixed Point Theorem to prove that $\mathcal{A}_4$ cannot act freely on any finite CW complex X such that the cohomology rings $H^*(X;\mathbb{Z})$ and $H^*(S^n \times S^n;\mathbb{Z})$ are isomorphic ([15, Theorem 2]). We can improve this statement in the following manner:

Proposition 3.2 ([5, Corollary 3.2]). *The alternating group $\mathcal{A}_4$ cannot act freely on any finite-dimensional CW complex X such that the cohomology rings $H^*(X;\mathbb{Z})$ and $H^*(S^n \times S^n;\mathbb{Z})$ are isomorphic, where n is any positive integer.*

Proof. Suppose that $\mathcal{A}_4$ acts freely on X as above. In view of Theorem 3.1, $H^n(X;\mathbb{Z})$ is a non-trivial $\mathcal{A}_4$-module. But $\mathcal{A}_4$ is generated by elements of order 3, so $H^n(X;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ is also a non-trivial $\mathbb{Z}_3$-module for some subgroup $\mathbb{Z}_3 \subseteq \mathcal{A}_4$. By Lemma 2.1, the $\mathbb{Z}_3$-module structure on $H^n(X;\mathbb{Z})$ comes from

$$\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix},$$

hence it does not split off a trivial direct summand. This contradicts Theorem 2.2. □

Remark 3.3. The finite-dimensionality hypothesis of Proposition 3.2 cannot be dropped: the product $E\mathcal{A}_4 \times (S^n)^k$ (k, n arbitrary) provides an example of an infinite-dimensional, free $\mathcal{A}_4$-space homotopy equivalent to $(S^n)^k$. (Here $E\mathcal{A}_4$ stands for the universal cover of the classifying space of $\mathcal{A}_4$.)

Theorem 3.4. *The alternating group $\mathcal{A}_4$ acts freely on $S^n \times S^n \times S^n$ if and only if $n = 1, 3, 7$.*

Proof. ($\Leftarrow$) As a preliminary remark, recall that $\mathcal{A}_4$ can be described twofolds: either by the presentation

$$\langle a, b \mid a^2 = b^3 = (ab)^3 = 1 \rangle,$$

or by the extension

$$0 \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to \mathcal{A}_4 \xrightarrow{\epsilon} \mathbb{Z}_3 \to 0.$$

We will make use of both.

Let $F_2 = \langle a, b \rangle$ be the free group on two generators. Define an F_2-action on $S^n \times S^n$ by setting

$$\begin{cases} a(x,y) = (-x, y) \\ b(x,y) = (y, y^{-1}x^{-1}) \end{cases} \quad \text{for } x, y \in S^n.$$

For $n = 1$ or 3, it is straightforward to verify that this action is trivial while restricted to the normal closure of a^2, b^3 and $(ab)^3$, and thus induces an $\mathcal{A}_4$-action on $S^n \times S^n$, with the subgroup $\mathbb{Z}_2 \oplus \mathbb{Z}_2 = \langle a, bab^2 \rangle \subseteq \mathcal{A}_4$ acting freely. The same statement is true for $n = 7$: the octonions form an alternative algebra, hence even though their multiplication is not associative in general, it is associative on any two-generated subalgebra ([19, Appendix A, Theorem 4.16]).

Now take any free action of $\mathbb{Z}_3$ on S^n (for example the one generated by the rotation $x \mapsto e^{2\pi i/3}x$, $x \in S^n$) and extend it to an action of $\mathcal{A}_4$ by means of the epimorphism ϵ. One easily checks that the product of these two actions gives rise to a free $\mathcal{A}_4$-action on $S^n \times S^n \times S^n$.

This construction should be attributed to Plakhta (cf. [16, Example 1]).

($\Rightarrow$) Suppose that $\mathcal{A}_4$ acts freely on $S^n \times S^n \times S^n$. In view of Theorem 3.1, $\mathcal{H} = H^n(S^n \times S^n \times S^n; \mathbb{Z})$ is a non-trivial $\mathcal{A}_4$-module. The only non-trivial normal subgroup of $\mathcal{A}_4$ is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, hence $\mathcal{H}$ is also a non-trivial $\mathbb{Z}_3$-module for any subgroup $\mathbb{Z}_3 \subseteq \mathcal{A}_4$. Since the $\mathbb{Z}_3$-module structure on $\mathcal{H}$ comes from a free $\mathbb{Z}_3$-action, $\mathcal{H}$ splits off a trivial direct summand by Theorem 2.2. Consequently, by Lemma 2.1, there exists a basis of $\mathcal{H}$ in which its $\mathbb{Z}_3$-module structure is given by

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now express the $\mathbb{Z}_3$-module structure on $\mathcal{H}$ by a matrix in the usual basis. After reducing modulo 2, the resulting matrix will be conjugate to

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

as such, it cannot be a permutation matrix. It now follows from Proposition 2.3 that $n = 1, 3, 7$. $\square$

Proposition 3.2 together with Theorem 3.4 show that $k(\mathcal{A}_4) = 3$.

Example 3.5. Let us describe another way of seeing that $\mathcal{A}_4$ acts freely on $S^1 \times S^1 \times S^1$. In view of Proposition 2.4, it suffices to produce a group extension

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \to \pi_1(N) \to \mathcal{A}_4 \to 1,$$

where N is a closed 3-dimensional aspherical manifold.

Think of S^1 as the additive group of real numbers modulo 1. Let N be the torus bundle over S^1 given by the mapping torus of the homeomorphism $h \colon S^1 \times S^1 \to S^1 \times S^1$, $h(t_1, t_2) = (-t_2, t_1 - t_2)$ for any t_1, $t_2 \in S^1$. (N is the manifold (1.5) of [13].) It is well-known that N is a closed 3-dimensional manifold, and its asphericity follows from the long exact sequence of homotopy groups of the corresponding fiber bundle. Furthermore,

$$\pi_1(N) \cong (\mathbb{Z} \oplus \mathbb{Z}) \rtimes_{h_*} \mathbb{Z} \cong \langle a, b, c \mid [a, b] = 1,\ cac^{-1} = b,\ cbc^{-1} = a^{-1}b^{-1} \rangle.$$

Let $\mathcal{Z} \subseteq \pi_1(N)$ be the subgroup generated by a^2, b^2 and c^3. Since c^3 commutes with both a and b, it is straightforward to see that $\mathcal{Z}$ is a normal subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, and that the quotient $\pi_1(N)/\mathcal{Z}$ is isomorphic to $\mathcal{A}_4$.

This approach makes it clear that the orbit space $S^1 \times S^1 \times S^1/\mathcal{A}_4$ of the arising $\mathcal{A}_4$-action is homeomorphic to N.

Remark 3.6. After passing the threshold, there is much more flexibility: $\mathcal{A}_4$ acts freely on $(S^{2n-1})^4$ for any integer $n \geq 1$. To see this, define an F_2-action on $S^n \times S^n \times S^n$ by setting

$$\begin{cases} a(x,y,z) = (-x,y,-z) \\ b(x,y,z) = (y,z,x) \end{cases} \quad \text{for } x, y, z \in S^n$$

and proceed as in the proof of Theorem 3.4.

Note that there is no point in considering free $\mathcal{A}_4$-action on products of even-dimensional spheres: if a finite group G acts freely on $X = S^{2n_1} \times S^{2n_2} \times \cdots \times S^{2n_k}$, then G is a 2-group because of the equality $2^k = \chi(X) = |G| \cdot \chi(X/G)$. Here χ denotes the Euler characteristic.

The reader interested in free actions of arbitrary alternating groups on products of equidimensional spheres is invited to consult [5].

4 Free $\mathcal{A}_4$-actions on $S^m \times S^n$

The requirement of equidimensionality of spheres in the product is actually crucial for Proposition 3.2 to hold.

Example 4.1 ([15, p. 543])**.** We will show that $\mathcal{A}_4$ acts freely on $S^2 \times S^3$. Write $SO(n)$ for the special orthogonal group of degree n. Consider the twisted product $SO(3) \times_{S^1} S^3$, with $S^1 \cong SO(2)$ acting as a subgroup on both $SO(3)$ and S^3. This, as usual, is a fiber bundle over $SO(3)/SO(2) \approx S^2$, with fiber S^3 and structure group S^1.

Observe that the S^1-action on S^3 is contained in the group action of S^3, and consequently $SO(3) \times_{S^1} S^3$ can be thought of as a principal S^3-bundle. Since

$$\pi_2(BS^3) \cong \pi_1(S^3) = 0,$$

the bundle is trivial, thus $SO(3) \times_{S^1} S^3 \approx S^2 \times S^3$. The conclusion follows from the fact that $\mathcal{A}_4$ is a subgroup of $SO(3)$.

It would be interesting to determine all pairs (m,n) for which there exists a free $\mathcal{A}_4$-action on $S^m \times S^n$. Let us summarize what is known in this direction:

- By Proposition 3.2, $m \neq n$.
- It follows from the discussion included in Remark 3.6 that m or n has to be odd.
- We will prove in Proposition 4.3 that $\mathcal{A}_4$ cannot act freely on $S^1 \times S^n$ for any $n \geq 1$.

As for the existence results:

- As explained in Example 4.1, $\mathcal{A}_4$ acts freely on $S^2 \times S^3$.
- Using the notion of fixity of a group, Adem–Davis–Ünlü proved that $\mathcal{A}_4$ acts freely on $S^{2n-1} \times S^{4n-5}$ for any $n \geq 3$ (see [2, Theorem 3.1]).

In order to prove that $\mathcal{A}_4$ cannot act freely on $S^1 \times S^n$ for any $n \geq 1$, we need the following basic lemma.

Lemma 4.2 ([12, Lemma 2.7]). *Let $0 \to A' \to A \xrightarrow{\kappa} A'' \to 0$ be a central extension of groups. If A' is a torsionfree abelian group and A'' is a torsion abelian group, then A is an abelian group.*

Proof. Let $a, b \in A$. Clearly, $\kappa(a)^n = 0$ for some $n > 0$ and, consequently, $a^n \in A'$. Thus $[a^n, b] = [a, b]^n = 1$. But $[a, b] \in A'$, which is torsionfree, so the conslusion follows. $\square$

Proposition 4.3. *The alternating group $\mathcal{A}_4$ cannot act freely on $S^1 \times S^n$ for any $n \geq 1$.*

Proof. In view of Proposition 3.2, we can assume without loss of generality that $n \geq 2$. If $\mathcal{A}_4$ acted freely on $S^1 \times S^n$, then by the theory of covering spaces, $\Gamma = \pi_1(S^1 \times S^n/\mathcal{A}_4)$ would act properly discontinuously on $S^n \times \mathbb{R}$. By [7, Lemma 4.2], a necessary condition for this to be possible is periodicity of Farrell cohomology of Γ. This in turn is equivalent to Γ having elementary abelian subgroups of rank at most 1 (see [6, Chapter X, Theorem 6.7]). We will show, however, that Γ contains $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ as a subgroup.

Consider the following commutative diagram which arises from the long exact sequence of homotopy groups of the covering $S^1 \times S^n \to S^1 \times S^n/\mathcal{A}_4$:

$$\begin{tikzcd}
& & 0 & 0 & \\
& & \mathbb{Z}_3 \arrow[u] \arrow[r, equal] & \mathbb{Z}_3 \arrow[u] & \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \Gamma \arrow[r] \arrow[u] & \mathcal{A}_4 \arrow[r] \arrow[u] & 1 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[u, equal] & \Gamma' \arrow[r] \arrow[u] & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \arrow[r] \arrow[u] & 0 \\
& & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

The top horizontal extension is central, hence the same is true for the bottom one. By Lemma 4.2, Γ' is abelian, and therefore it suffices to find two elements of order 2 in Γ' to conclude the proof. In order to do so, choose a copy of $\mathbb{Z}_2 \subseteq \mathbb{Z}_2 \oplus \mathbb{Z}_2$ to obtain:

$$\begin{tikzcd}
& & 0 & 0 & \\
& & \mathbb{Z}_2 \arrow[u] \arrow[r, equal] & \mathbb{Z}_2 \arrow[u] & \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \Gamma' \arrow[r] \arrow[u] & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \arrow[r] \arrow[u] & 1 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[u, equal] & \Gamma'' \arrow[r] \arrow[u] & \mathbb{Z}_2 \arrow[r] \arrow[u] & 0 \\
& & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

The bottom horizontal extension is *a fortiori* central, hence Γ'' is either $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}_2$. If the first possibility holds, then Γ' is $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}_2$ (the only other extension of $\mathbb{Z}_2$ by $\mathbb{Z}$, the infinite dihedral group, is non-abelian). Both these choices imply that Γ is abelian, which is impossible. Thus $\Gamma'' \cong \mathbb{Z} \oplus \mathbb{Z}_2$, and consequently Γ' contains an element of order 2 for every copy of $\mathbb{Z}_2 \subseteq \mathbb{Z}_2 \oplus \mathbb{Z}_2$. □

Remark 4.4. (1) The above argument works equally well if $\mathcal{A}_4$ is replaced with any finite nonabelian group which contains $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ as a normal subgroup and does admit an epimorphism onto $\mathbb{Z}_2$.

(2) See [11, Corollary 2.3] for an alternative proof of Proposition 4.3 for n even.

Remark 4.5. Propositions 3.2 and 4.3 show that the action presented in Example 4.1 gives the lowest-dimensional possibility, i.e., if $\mathcal{A}_4$ acts freely on $S^m \times S^n$, then $m + n \geq 5$.

It is also worth mentioning that any free $\mathcal{A}_4$-action on $S^m \times S^n$ has to be exotic, in the sense that it cannot come from a product of two actions on single spheres. For if $\mathcal{A}_4$ acted freely on $S^m \times S^n$ via a product action, then by taking an appropriate number of joins of each sphere (recall that a k-fold join of an n-sphere is a $(k(n+1)-1)$-sphere), one would obtain a free $\mathcal{A}_4$-action on $S^{(m+1)(n+1)-1} \times S^{(m+1)(n+1)-1}$, which contradicts Proposition 3.2. This was first observed by Adem–Smith ([3, Theorem 5.1]).

5 Free $\mathcal{S}_3$-actions on $S^m \times S^n$

Let us back up a little and look at free actions of the symmetric group $\mathcal{S}_3$. The story starts with Milnor, who proved that $\mathcal{S}_3$ cannot act freely on any sphere ([14, Corollary 1]). On the other hand, Swan constructed a finite, 3-dimensional CW complex homotopy equivalent to S^3 which admits a free $\mathcal{S}_3$-action ([17, Appendix]). Thus $k(\mathcal{S}_3) = 1$, but it is nevertheless worthwhile to inquire about actions on actual products spheres. We have:

Proposition 5.1. *The symmetric group $\mathcal{S}_3$ acts freely on $S^m \times S^n$ if and only if m or n is odd. In particular, $\mathcal{S}_3$ acts freely on $S^n \times S^n$ if and only if n is odd.*

Proof. Because of an Euler characteristic argument (see Remark 3.6), it suffices to construct a free $\mathcal{S}_3$-action on $S^m \times S^n$ whenever m or n is odd.

Assume that m is odd, and think of S^m as a subspace of $\mathbb{C}^{(m+1)/2}$. We will proceed similarly as in the proof of Theorem 3.4. Let $F_2 = \langle a, b \rangle$ be the free group on two generators. Define an F_2-action on S^m by

$$\begin{cases} ax = \bar{x} \\ bx = e^{2\pi i/3}x \end{cases} \quad \text{for } x \in S^m.$$

Since $\mathcal{S}_3$ can be presented as $\langle a, b \mid a^2 = b^3 = (ab)^2 = 1 \rangle$, it is straightforward to verify that this action induces an $\mathcal{S}_3$-action on S^m, which clearly is free while restricted to the subgroup $\mathbb{Z}_3 = \langle b \rangle \subseteq \mathcal{S}_3$.

Now consider the antipodal action on S^n and extend it to an $\mathcal{S}_3$-action via the epimorphism ϵ coming from the extension $0 \to \mathbb{Z}_3 \to \mathcal{S}_3 \xrightarrow{\epsilon} \mathbb{Z}_2 \to 0$. The product of these two actions gives rise to a free $\mathcal{S}_3$-action on $S^m \times S^n$. □

Remark 5.2. For any $n \geq 1$, the orbit space $S^1 \times S^n/\mathcal{S}_3$ of the action constructed in the proof of Proposition 5.1 is homeomorphic to the connected sum $\mathbb{R}P^{n+1}\#\mathbb{R}P^{n+1}$ of projective spaces. Indeed,

$$S^1 \times S^n/\mathcal{S}_3 \approx (S^1 \times S^n/\mathbb{Z}_3)/\mathbb{Z}_2 \approx \big((S^1/\mathbb{Z}_3) \times S^n\big)/\mathbb{Z}_2 \approx S^1 \times S^n/\mathbb{Z}_2 \\ \approx \mathbb{R}P^{n+1}\#\mathbb{R}P^{n+1},$$

because the last $\mathbb{Z}_2$-action on $S^1 \times S^n$ is given by $(x, y) \mapsto (\bar{x}, -y)$ for $x \in S^1$, $y \in S^n$.

If $n = 1$ or 2, the same statement is true for an arbitrary free $\mathcal{S}_3$-action on $S^1 \times S^n$. This is clear for $n = 1$; for $n = 2$, it is a consequence of [18, Corollary 2]. In general, if n is even, it follows from [11, Corollary 2.3] that $S^1 \times S^n/\mathcal{S}_3$ is homotopy equivalent to $\mathbb{R}P^{n+1}\#\mathbb{R}P^{n+1}$ for any free $\mathcal{S}_3$-action.

Remark 5.3. The argument of Proposition 5.1 can be applied, *mutatis mutandis*, to produce a free action of any dihedral group on $S^m \times S^n$ provided m or n is odd.

In general, $k(\mathcal{A}_d) \leq k(\mathcal{S}_d) \leq 2k(\mathcal{A}_d) + 1$ for any $d \geq 1$. The second inequality follows from the "coinduction" presented in the proof of Proposition 1.1 and the "piecewise" method of building actions, as given in the proofs of Theorem 3.4 and Proposition 5.1. Apart from that, not much else can be said about the number k for the class of symmetric groups.

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Koszul complexes and Chevalley's theorems for Lie algebroids

We use Koszul complexes and Chevalley-type theorems to calculate the cohomology $\mathbf{H}(A)$ of a transitive Lie algebroid A under some assumptions on the isotropy Lie algebras.

1 Introduction

How can we calculate the cohomology $\mathbf{H}(A)$ of a transitive Lie algebroid A with the Atiyah sequence $0 \longrightarrow \boldsymbol{g} \hookrightarrow A \xrightarrow{\#_A} TM \longrightarrow 0$ **?** This is one of the fundamental questions for the topology of Lie algebroids [I-K-V], [M]. A classical method is to use spectral sequences. We can use the Leray spectral sequence for the Čech–de Rham complex of transitive Lie algebroids [K-M-1] as well as the Hochschild-Serre spectral sequence for the pair of Lie algebras $(\operatorname{Sec}\boldsymbol{g}, \operatorname{Sec} A)$ and the observation that the vector bundle of the cohomology $\mathbf{H}(\boldsymbol{g})$ of the isotropy Lie algebras, $\mathbf{H}(\boldsymbol{g})_{|x} = \mathbf{H}\left(\boldsymbol{g}_{|x}\right)$, is flat, and that $E_2^{j,i} = \mathbf{H}^j_{\nabla}\left(M; \mathbf{H}^i(\boldsymbol{g})\right)$ where ∇ is the flat covariant derivative in $\mathbf{H}^i(\boldsymbol{g})$ [K-M-2], [K-M-3].

In this paper we propose an adaptation of the method of Koszul complexes and Chevalley-type theorems [G-H-V, Vol. III] to the calculation of $\mathbf{H}(A)$. Originally the method is based on the operation of a reductive Lie algebra in a graded differential algebra admitting an algebraic connection. A fundamental theorem of Chevalley gives a homomorphism from the corresponding Koszul complex which induces an isomorphism of cohomology. Classically, this isomorphism is applied to the cohomology of principal fibre bundles. Namely: the Chevalley theorem (for pfb's) says that under some assumptions, the cohomology of the total space $\mathbf{H}(P)$ of a pfb P depends uniquely on the cohomology of the base manifold M and the characteristic classes (the Chern-Weil homomorphism $h_P : \left(\bigvee \mathfrak{g}^*\right)_{I_G} \longrightarrow \mathbf{H}(M)$). It turns out that this assertion has a counterpart for Lie algebroids, but in this context we cannot use the standard operation of a Lie algebra directly. We propose some modification of this method.

2 Lie algebroid of a principal fibre bundle, Lie functor

2.1 Examples of Lie algebroids

2.1.1 Lie algebroid of a Lie group

The Lie algebroid of a Lie group G (the infinitesimal object of a Lie group G) is simply its Lie algebra $\mathfrak{g} = T_eG = TG/G$ (for example, through the right action of G on TG we obtain the "right Lie algebra of a Lie group").

2.1.2 Lie algebroid of a principal fibre bundle

The vector space $\mathbf{A}(P) := TP/G$ of cosets of the right action of G on TP (introduced by M. Atiyah in 1955) is an infinitesimal object of a principal fibre bundle $P(M,G)$. It has two extra structures: a Lie bracket in the space of global cross-sections $\operatorname{Sec}\mathbf{A}(P)$ and a linear homomorphism $\#_{\mathbf{A}(P)} : \mathbf{A}(P) \longrightarrow TM$ called the anchor. The Lie bracket in $\operatorname{Sec}\mathbf{A}(P)$ is introduced via the isomorphism $\operatorname{Sec}(\mathbf{A}(P)) \cong \mathfrak{X}^R(P)$ where $\mathfrak{X}^R(P)$ is the space of right invariant vector fields on P with the usual Lie bracket. The anchor is defined by $\#_{\mathbf{A}(P)} : \mathbf{A}(P) \longrightarrow TM$, $[v] \longmapsto \pi_*(v)$ where $\pi : P \to M$ is the projection of P. The anchor $\#_{\mathbf{A}(P)}$ is bracket-preserving: $\#_{\mathbf{A}(P)}([\![\xi_1,\xi_2]\!]) = \left[\#_{\mathbf{A}(P)}(\xi_1), \#_{\mathbf{A}(P)}(\xi_2)\right]$, and the Leibniz formula holds: $[\![\xi_1, f\cdot\xi_2]\!] = f\cdot[\![\xi_1,\xi_2]\!] + \left(\#_{\mathbf{A}(P)}(\xi_1)\right)(f)\cdot\xi_2$. The Lie algebroid of the trivial principal fibre bundle $P = M\times G$ is equal to

$$\mathbf{A}(P) = TP/G = T(M\times G)/G = TM\times(TG/G) = TM\times\mathfrak{g},$$

with the bracket $[\![(X,\sigma),(Y,\eta)]\!] = ([X,Y], X(\eta) - Y(\sigma) + [\sigma,\eta])$ for $X, Y \in \mathfrak{X}(M)$, $\sigma, \eta \in C^{\infty}(M, \mathfrak{g})$, and the anchor $\#_{TM\times\mathfrak{g}} = pr_1 : TM \times \mathfrak{g} \to TM$.

2.2 Pradines' definition of a Lie algebroid

Generalizing the structure $\left(\mathbf{A}(P),\ [\![\cdot,\cdot]\!], \#_{A(P)}\right)$ for a pfb $P(M,G)$ J. Pradines gives the definition of a Lie algebroid [P]:

Definition 1. *A Lie algebroid on a manifold M is a triple $(A, [\![\cdot,\cdot]\!], \#_A)$ where A is a vector bundle on M, $(\mathrm{Sec}\, A, [\![\cdot,\cdot]\!])$ is an $\mathbb{R}$-Lie algebra, $\#_A : A \to TM$ is a linear homomorphism of vector bundles and the following Leibniz condition is satisfied:*

$$[\![\xi, f\cdot\eta]\!] = f \cdot [\![\xi,\eta]\!] + \gamma_L(\xi)(f)\cdot\eta, \quad f \in C^{\infty}(M),\ \xi,\eta \in \mathrm{Sec}\, A.$$

The anchor is bracket-preserving, $\#_A \circ [\![\xi,\eta]\!] = [\#_A \circ \xi, \#_A \circ \eta]$.

The image of the anchor, $\mathrm{Im}\, \#_A \subset TM$, is an integrable non-constant-rank (in general) distribution whose leaves form a Stefan foliation of M. If the anchor $\#_A$ is of constant rank then the Lie algebroid A is called regular and $\mathrm{Im}\,\#_A$ forms a regular foliation on M. The Lie algebroid is called transitive if $\#_A$ is an epimorphism. A transitive Lie algebroid is called integrable if it is isomorphic to the Lie algebroid of a principal fibre bundle.

We deal here only with transitive Lie algebroids.

For a transitive Lie algebroid A we have the Atiyah sequence

$$0 \longrightarrow \boldsymbol{g} \hookrightarrow A \xrightarrow{\#_A} TM \longrightarrow 0.$$

The vector bundle $\boldsymbol{g}$ is a Lie algebra bundle, called the adjoint of A; in particular, all the isotropy Lie algebras $\boldsymbol{g}_{|x}$ are isomorphic.

Example 2. *(1) A single Lie algebra $\mathfrak{g}$ is a Lie algebroid over a one–point set and with the zero anchor.*

(2) The tangent bundle TM of a manifold M is a Lie algebroid on M with id_{TM} as anchor and with the usual Lie bracket of vector fields.

(3) Trivial Lie algebroid: $TM \times \mathfrak{g}$ with the projection pr_1 as anchor and with the bracket given by

$$[\![\,(X,\sigma),(Y,\eta)\,]\!] = ([X,Y], X(\eta) - Y(\sigma) + [\sigma,\eta]),$$

$X, Y \in \mathfrak{X}(M)$, $\sigma, \eta \in C^{\infty}(M; \mathfrak{g})$, *is a transitive Lie algebroid, called trivial. (Each transitive Lie algebroid L over a contractible manifold is isomorphic to the trivial one).*

(4) The Lie algebroid $A(P) = TP/G$ *of a* G*-principal fibre bundle* $P = P(M, G)$.

(5) The Lie algebroid $\mathbf{A}(\mathfrak{f})$ *of a vector bundle* $\mathfrak{f}$*: With a vector bundle* $\mathfrak{f}$ *we associate a transitive Lie algebroid* $\mathbf{A}(\mathfrak{f})$ *(isomorphic to the Lie algebroid of the principal fibre bundle of all frames of* $\mathfrak{f}$, $\mathbf{A}(\mathfrak{f}) = \mathbf{A}(\mathrm{L}\,\mathfrak{f})$*) whose space of global cross-sections* $\operatorname{Sec}\mathbf{A}(\mathfrak{f})$ *is equal to the space of all covariant differential operators for* $\mathfrak{f}$*. The Lie algebra bundle adjoint to* $\mathbf{A}(\mathfrak{f})$ *is equal to* $\operatorname{End}(\mathfrak{f})$*, so the Atiyah sequence reads*

$$0 \longrightarrow \operatorname{End}(\mathfrak{f}) \longrightarrow \mathbf{A}(\mathfrak{f}) \longrightarrow TM \to 0.$$

Example 3 (Other examples). *(6) The Lie algebroid* $A(M, \mathcal{F})$ *of a transversally complete foliation* $(M, \mathcal{F})$ *of a connected Hausdorff paracompact manifold* M*, in particular:*

(6') The Lie algebroid $A(G; H)$ *of a nonclosed Lie subgroup* H *of* G*: It is the Lie algebroid of the TC-foliation* $\mathcal{F}_{G,H} = \{aH; a \in G\}$ *of left cosets of a nonclosed Lie subgroup* H *in a Lie group* G*. These include nonintegrable Lie algebroids.*

(7) Poisson manifolds yield nontransitive Lie algebroids.

Definition 4. *By a homomorphism of Lie algebroids* $F : (A, [\![\cdot,\cdot]\!], \#_A) \longrightarrow (A', [\![\cdot,\cdot]\!], \#_{A'})$ *on a manifold* M *we mean a linear homomorphism* $F : A \to A'$ *of vector bundles commuting with the anchors:*

$$\begin{array}{ccc} A & \xrightarrow{F} & A' \\ \downarrow \#_A & & \downarrow \#_{A'} \\ TM & = & TM \end{array}$$

and such that F *is a homomorphism of the Lie algebras of global cross-sections:*

$$F([\![\xi_1, \xi_2]\!]) = [\![F\xi_1, F\xi_2]\!], \quad \xi_i \in \operatorname{Sec} A.$$

A homomorphism $F : A \longrightarrow B$ *of transitive Lie algebroids induces a linear homomorphism of the adjoint Lie algebra bundles* $F^+ : \boldsymbol{g} \longrightarrow \boldsymbol{g}'$ *and for any* $x \in M$, $F_x^+ : \boldsymbol{g}_{|x} \longrightarrow \boldsymbol{g}'_{|x}$ *is a homomorphism of Lie algebras.*

We obtain in this way a homomorphism of Atiyah sequences,

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ \boldsymbol{g} & \xrightarrow{F^{+}} & \boldsymbol{g}' \\ \downarrow & & \downarrow \\ A & \xrightarrow{F} & A' \\ \downarrow & & \downarrow \\ TM & = & TM \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

2.3 Lie functor

To have a Lie functor for pfb's we need to define a homomorphism of Lie algebroids induced by a homomorphism of pfb's. Let P and P' be two pfbs with structural Lie groups G and G', respectively. Assume that $\mu : G \longrightarrow G'$ is a homomorphism of Lie groups, and $F : P \longrightarrow P'$ a μ-homomorphism of pfbs, i.e. $F(z \cdot a) = F(z) \cdot a'$. Then the linear homomorphism (the Lie algebroid differential of F)

$$F_{*} : \mathbf{A}(P) \longrightarrow \mathbf{A}(P'), \quad [v_z] \longmapsto [dF_{*z}(v_z)],$$

is a homomorphism of the induced Lie algebroids.

2.4 Cohomology of a Lie algebroid

To a Lie algebroid A we associate the cohomology algebra $\mathbf{H}(A)$ defined via the DG-algebra of A-differential forms (with real coefficients) $(\Omega(A), d_A)$, where

$$\begin{aligned} \Omega(A) &= \operatorname{Sec} \bigwedge A^{*}, \\ d_A &: \Omega^{*}(A) \longrightarrow \Omega^{*+1}(A) \end{aligned}$$

$$\begin{aligned} (d_A z)(\xi_0, ..., \xi_k) = & \sum_{j=0}^{k} (-1)^j \, (\#_A \circ \xi_j)\,(z(\xi_0, ...\hat{\jmath}..., \xi_k)) \\ & + \sum_{i<j} (-1)^{i+j} \, z([\![\xi_i, \xi_j]\!], \xi_0, ...\hat{\imath}...\hat{\jmath}..., \xi_k), \end{aligned}$$

$z \in \Omega^k(A)$, $\xi_i \in \operatorname{Sec} A$. The exterior derivative d_A induces the cohomology algebra

$$\mathbf{H}(A) = \mathbf{H}(\Omega(A), d_A).$$

Why the differential d_A must be given by the above formula? It is easy to obtain this formula starting with the Lie algebroid of a Lie groupoid $\Phi = (\Phi, (\alpha, \beta), \cdot . M)$ on a manifold M, with source α and target β and partial multiplication $\cdot$. Let $i : M \to \Phi$ be the embedding of M onto the submanifold of units, $i(x) = u_x$, of this Lie groupoid. Then

$$\mathbf{A}(\Phi) = i^*(T^\alpha \Phi)$$

where $T^\alpha \Phi$ is the subbundle of α-vertical vectors. We see that for any $x \in M$, the submanifold $\Phi_x = \alpha^{-1}(x)$ of all elements starting at x (i.e. having x as source) forms a Φ_x^x-pfb (Φ_x^x is the Lie isotropy group at x, $\Phi_x^x = \{h \in \Phi : \alpha h = \beta h = x\}$) with the projection $\beta_x : \Phi_x \longrightarrow M$. We have $\mathbf{A}(\Phi)_{|x} = T_{u_x}(\Phi_x)$, the tangent space to the total space Φ_x at the unit x. For all pfb's Φ_x we can consider standard differential operators, like the exterior derivative of usual differential forms (or Lie derivative and substitution operator), and pass to the units u_x and "glue". By this procedure we obtain just d_A.

Example 5. *(1) If $A = \mathbf{A}(P) = TP/G$ for a G-principal fibre bundle $P \longrightarrow M$ then*

$$\Omega(A) \cong \Omega^R(P) \hookrightarrow \Omega(P),$$

$\Omega^R(P)$ are G-right invariant differential forms on P and

$$\mathbf{H}(A) \cong \mathbf{H}\left(\Omega^R(P)\right) \xrightarrow{i} \mathbf{H}_{dR}(P).$$

The homomorphism i is an isomorphism if G is compact and connected.

(2) If $A = A(M; \mathcal{F}) \longrightarrow W$ is the Lie algebroid of a TC-foliation $\mathcal{F}$ on M (W is the so called basic manifold of the foliation $\mathcal{F}$), then [K3, Th. 6.2]

$$\Omega(A) \cong \Omega_b(M; \mathcal{F}),$$

$\Omega_b(M; \mathcal{F})$ is the algebra of $\mathcal{F}$-basic differential forms, therefore $\mathbf{H}(A) \cong \mathbf{H}_b(M; \mathcal{F})$ is the algebra of basic cohomology.

Below, we will propose a calculation of $\mathbf{H}(A)$ using the old technique of Koszul complexes and the so-called Chevalley theorems known for principal fibre bundles with structural Lie groups with reductive Lie algebras.

These Chevalley theorems (for pfb's) say that under some assumptions, the cohomology of the total space $\mathbf{H}(P)$ of a pfb P depends uniquely on the cohomology of the base manifold M and the characteristic classes (the Chern-Weil homomorphism $h_P : \left(\bigvee \mathfrak{g}^*\right)_{I_G} \longrightarrow \mathbf{H}(M)$). It turns out that this assertion has a counterpart for Lie algebroids.

3 Koszul complexes and Chevalley's theorem in the framework of Lie algebroids

3.1 Representations of Lie algebroids and invariant cross-sections

Consider an arbitrary transitive Lie algebroid A on a manifold M with the Atiyah sequence $0 \longrightarrow \boldsymbol{g} \longrightarrow A \overset{\#_A}{\longrightarrow} TM \longrightarrow 0$ and a vector bundle $\mathfrak{f}$ on M.

Definition 6. *By a representation of A on $\mathfrak{f}$ we mean a homomorphism of Lie algebroids*

$$T : A \longrightarrow \mathbf{A}(\mathfrak{f}).$$

Look at the induced homomorphism of Atiyah sequences:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\boldsymbol{g} & \overset{T^+}{\longrightarrow} & \operatorname{End}(\mathfrak{f}) \\
\downarrow & & \downarrow \\
A & \overset{T}{\longrightarrow} & \mathbf{A}(\mathfrak{f}) \\
\downarrow & & \downarrow \\
TM & = & TM \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

At each point x we get a representation of the isotropy Lie algebra $\boldsymbol{g}_{|x}$ on the vector space $\mathfrak{f}_{|x}$,

$$T_x^+ : \boldsymbol{g}_{|x} \longrightarrow \operatorname{End}\left(\mathfrak{f}_{|x}\right).$$

For a cross-section $\xi \in \operatorname{Sec} A$ its image $T\xi \in \operatorname{Sec} \mathbf{A}(\mathfrak{f})$ determines a covariant differential operator

$$\mathcal{L}_{T\xi} : \operatorname{Sec} \mathfrak{f} \longrightarrow \operatorname{Sec} \mathfrak{f}.$$

Example 7. *(The representation of a Lie algebroid induced by a representation of a pfb) Let G be any Lie group and $\mu : G \longrightarrow GL(V)$ be any representation of G on a vector space V. If $F : P \longrightarrow \mathrm{L}\mathfrak{f}$ is a μ-homomorphism of pfb's (called a μ-representation of P on $\mathfrak{f}$) then its Lie algebroid's differential $F_* : \mathbf{A}(P) \longrightarrow \mathbf{A}(\mathfrak{f})$ is a representation of $\mathbf{A}(P)$ on $\mathfrak{f}$.*

Definition 8. *A cross-section $\nu \in \operatorname{Sec}\mathfrak{f}$ is called T-invariant (or T-parallel) if it belongs to the kernel of $\mathcal{L}_{T\xi}$ for each ξ, i.e.*

$$\mathcal{L}_{T\xi}(\nu) = 0 \quad \text{for all} \ \ \xi \in \operatorname{Sec} A.$$

The space of all T-invariant cross-sections is denoted by $(\operatorname{Sec}\mathfrak{f})_{I_T}$. If $\nu \in \operatorname{Sec}\mathfrak{f}$ is invariant then its value ν_x at x is invariant with respect to $T^+_{|x} : \boldsymbol{g}_{|x} \to \operatorname{End}\left(\mathfrak{f}_{|x}\right)$, i.e.

$$\nu_x \in \left(\mathfrak{f}_{|x}\right)_{I_{T^+_x}}.$$

One can prove that for each transitive Lie algebroid A and each representation $T : A \longrightarrow \mathbf{A}(\mathfrak{f})$ the following theorem holds.

Theorem 9. *If ν_1 and ν_2 are T-invariant cross-sections of $\mathfrak{f}$ and they are equal at some point $x_0 \in M$, $\nu_1(x_0) = \nu_2(x_0)$, then they are equal globally, $\nu_1 = \nu_2$ (M is assumed to be connected), see [M], [K2].*

Therefore, the evaluation map

$$(\operatorname{Sec}\mathfrak{f})_{I_T} \longrightarrow \left(\mathfrak{f}_{|x}\right)_{I_{T^+_x}}, \quad \nu \longmapsto \nu(x),$$

is a monomorphism. Denote its image by

$$\left(\widetilde{\mathfrak{f}_{|x}}\right)_{I_{T^+_x}};$$

it contains all invariant vectors $u \in \left(\mathfrak{f}_{|x}\right)_{I_{T^+_x}}$ which can be extended to globally defined invariant cross-sections, i.e.

$$(\operatorname{Sec}\mathfrak{f})_{I_T} \cong \left(\widetilde{\mathfrak{f}_{|x}}\right)_{I_{T^+_x}} \subset \left(\mathfrak{f}_{|x}\right)_{I_{T^+_x}}.$$

Moreover, each invariant vector $u \in \left(\mathfrak{f}_{|x}\right)_{I^o\left(T^+_x\right)}$ can be extended to a locally defined (on some neighbourhood of x) invariant cross-section of the vector bundle $\mathfrak{f}$.

There is a wider class of Lie algebroids (integrable and nonintegrable) and representations where each invariant vector $u \in \left(\mathfrak{f}_{|x}\right)_{I_{T_x^+}}$ can be extended to globally defined invariant cross-sections.

Theorem 10 ([K1]). *Let P be a connected G-principal fibre bundle (G can be disconnected) and let $F : P \longrightarrow \mathrm{L}\,\mathfrak{f}$ be any μ-representation of P on $\mathfrak{f}$ where $\mu : G \longrightarrow GL(V)$ is a representation of G on V. Denote by $\mu_* : \mathfrak{g} \longrightarrow \mathrm{End}(V)$ the differential of μ (it is a representation of the Lie algebra $\mathfrak{g}$ of G on V). Then for the induced representation $F_* : \mathbf{A}(P) \longrightarrow \mathbf{A}(\mathfrak{f})$ of the Lie algebroid $\mathbf{A}(P)$ on $\mathfrak{f}$ we have*

$$(\mathrm{Sec}\,\mathfrak{f})_{I_{F_*}} \cong V_{I(\mu)} \quad \subset \quad V_{I(\mu_*)} \cong \left(\mathfrak{f}_{|x}\right)_{I_{F_{*x}^+}} .$$

If additionally G is connected then each invariant vector $v \in \left(\mathfrak{f}_{|x}\right)_{I_{F_{*x}^+}}$ (with respect to the representation $F_{*|x}^+$) can be extended to a globally defined F_*-invariant cross-section of $\mathfrak{f}$ and

$$(\mathrm{Sec}\,\mathfrak{f})_{I_{F_*}} \cong V_{I(\mu)} = V_{I(\mu_*)} \cong \left(\mathfrak{f}_{|x}\right)_{I_{F_{*x}^+}} .$$

If G is not connected then there may be invariant vectors which sometimes extend to global cross-sections and sometimes not (the Pfaffian is a typical example).

A representation $T : A \longrightarrow \mathbf{A}(\mathfrak{f})$ extends to representations on the associated vector bundles such as the dual bundle $\mathfrak{f}^*$, the exterior and symmetric powers $\bigwedge \mathfrak{f}^*$, $\bigvee^l \mathfrak{f}^*$ and their tensor products $\bigwedge \mathfrak{f}^* \otimes \bigvee^l \mathfrak{f}^*$.

3.2 Weil algebra for Lie algebroids [K1]

A fundamental example of a representation is the adjoint representation of A on the adjoint Lie algebra bundle $\boldsymbol{g}$ defined by

$$\begin{aligned} ad_A &: A \longrightarrow \mathbf{A}(\boldsymbol{g}), \\ ad_A(\xi) &: \mathrm{Sec}\,\boldsymbol{g} \longrightarrow \mathrm{Sec}\,\boldsymbol{g}, \quad \nu \longmapsto [\![\xi, \nu]\!]. \end{aligned}$$

Clearly the induced representation at an arbitrary point x, $\left(ad_A^+\right)_{|x}$, is the adjoint representation of the Lie algebra $\boldsymbol{g}_{|x}$,

$$\left(ad_A^+\right)_{|x} = ad_{\boldsymbol{g}_{|x}} : \boldsymbol{g}_{|x} \longrightarrow \mathrm{End}\left(\boldsymbol{g}_{|x}\right).$$

The adjoint representation ad_A induces representations on the associated vector bundles $\bigwedge \boldsymbol{g}^*$, $\bigvee^l \boldsymbol{g}^*$ (the skew symmetric and symmetric powers of the dual bundle $\boldsymbol{g}^*$) and on

$$(W\boldsymbol{g})^{k,2l} := \bigwedge \boldsymbol{g}^* \otimes \bigvee^l \boldsymbol{g}^*,$$

denoted also by ad_A. Put

$$\begin{aligned}(\mathcal{W}\boldsymbol{g})^{k,2l} &= \operatorname{Sec}(W\boldsymbol{g})^{k,2l}, \\ \mathcal{W}\boldsymbol{g} &= \bigoplus\nolimits_{k,l} \operatorname{Sec}(W\boldsymbol{g})^{k,2l}.\end{aligned}$$

For a point $x \in M$ we take the anticommutative (bi)graded tensor product of anticommutative graded algebras, i.e. the Weil algebra of the space $\boldsymbol{g}_{|x}$,

$$\begin{aligned}&W\boldsymbol{g}_{|x} = \bigwedge \boldsymbol{g}^*_{|x} \bigotimes \bigvee \boldsymbol{g}^*_{|x}, \\ &W\boldsymbol{g}_{|x} = \bigoplus\nolimits_{k.l} \left(W\boldsymbol{g}_{|x}\right)^{k,2l}, \quad \left(W\boldsymbol{g}_{|x}\right)^{k,2l} = \bigwedge^k \boldsymbol{g}^*_{|x} \bigotimes \bigvee^l \boldsymbol{g}^*_{|x}.\end{aligned}$$

The module $\mathcal{W}\boldsymbol{g}$ is a bigraded algebra with multiplication defined pointwise, called the *Weil algebra of the Lie algebroid A.*

In the space $W\boldsymbol{g}_{|x} = \bigwedge \boldsymbol{g}^*_{|x} \bigotimes \bigvee \boldsymbol{g}^*_{|x}$ (as for an arbitrary Lie algebra) there exist three standard operators: the substitution operator, the differential, and the adjoint representation, here denoted by

$$(\iota_x)_\nu\,, \ \delta_{W_x}\,, \ (\theta_x)_\nu\,, \quad \nu \in \boldsymbol{g}_{|x}.$$

It is easy to see that the adjoint representation $\theta_x^{k,2l} : \boldsymbol{g}_{|x} \to \operatorname{End}\left(W\boldsymbol{g}_{|x}\right)^{k,2l}$ is induced by the adjoint representation ad_A of the Lie algebroid A on $(W\boldsymbol{g})^{k,2l} = \bigwedge^k \boldsymbol{g}^* \bigotimes \bigvee^l \boldsymbol{g}^*$, $k,l \geq 0$, at a point x.

We have:

(a) $(\iota_\nu)_x$ is an antiderivation of degree -1 defined by

$$(\iota_\nu)_x (\Phi \otimes \Gamma) = (\iota_\nu)_x \Phi \otimes \Gamma,$$

$\Phi \in \bigwedge \boldsymbol{g}^*_{|x}, \quad \Gamma \in \bigvee \boldsymbol{g}^*_{|x},$

(b) δ_{W_x} is an antiderivation of degree $+1$ defined by

$$\delta_{W_x}(h^*\otimes 1)=1\otimes h^*+\delta_{\boldsymbol{g}_{|x}}h^*\otimes 1,$$

where $h^*\in\boldsymbol{g}^*_{|x}$, $\delta_{\boldsymbol{g}_{|x}}$ is the Chevalley-Eilenberg differential

$$\delta_{W_x}(1\otimes h^*)\in(W\boldsymbol{g})^{1,2}=\boldsymbol{g}^*\otimes\boldsymbol{g}^*,$$

such that

$$(\iota_\nu)_x(\delta_{W_x}(1\otimes h^*))=(\theta_x)_\nu h^*.$$

The operators $(\iota_x)_\nu$, δ_{W_x}, $(\theta_x)_\nu$, $x\in M$, together give operators on smooth cross-sections

$$\iota_\nu,\ \delta_{\mathcal{W}},\ \theta_\nu:\mathcal{W}\boldsymbol{g}\longrightarrow\mathcal{W}\boldsymbol{g},\quad \nu\in\operatorname{Sec}\boldsymbol{g}.$$

The cross-section $\Theta\in\mathcal{W}\boldsymbol{g}$ is called horizontal if $\iota_\nu\Theta=0$ for all $\nu\in\operatorname{Sec}\boldsymbol{g}$. Denote by

$$(\mathcal{W}\boldsymbol{g})_\iota$$

the space of horizontal elements.

Lemma 11. *The space $(\mathcal{W}\boldsymbol{g})_\iota$ of horizontal elements is a subalgebra of the Weil algebra $\mathcal{W}\boldsymbol{g}$ and contains only symmetric tensors:*

$$(\mathcal{W}\boldsymbol{g})_\iota=\bigoplus\nolimits_l\operatorname{Sec}\bigvee\nolimits^l\boldsymbol{g}^*.$$

Denote the space of global cross-sections of the vector bundle

$$(W\boldsymbol{g})^{k,2l}=\bigwedge\nolimits^k\boldsymbol{g}^*\bigotimes\bigvee\nolimits^l\boldsymbol{g}^*$$

invariant with respect to the adjoint representation of A on $(W\boldsymbol{g})^{k,2l}$ (for brevity) by

$$(\mathcal{W}\boldsymbol{g})^{k,2l}_{I^o}\subset(\mathcal{W}\boldsymbol{g})^{k,2l}$$

and put

$$(\mathcal{W}\boldsymbol{g})_{I^o}=\bigoplus\nolimits_{k,l}(\mathcal{W}\boldsymbol{g})^{k,2l}_{I^o}\subset\mathcal{W}\boldsymbol{g}.$$

Proposition 12. *$(\mathcal{W}\boldsymbol{g})_{I^o}$ is a subalgebra of the Weil algebra $\mathcal{W}\boldsymbol{g}$. Denote by $(\mathcal{W}\boldsymbol{g})_{I^o,\iota}$ the subalgebra of invariant and horizontal elements of the Weil algebra $\mathcal{W}\boldsymbol{g}$. The operator $\delta_{\mathcal{W}}:\mathcal{W}\boldsymbol{g}\longrightarrow\mathcal{W}\boldsymbol{g}$ maps invariant elements of $\mathcal{W}\boldsymbol{g}$ into invariant ones defining an antiderivation*

$$\delta_{\mathcal{W},I^o}:(\mathcal{W}\boldsymbol{g})_{I^o}\longrightarrow(\mathcal{W}\boldsymbol{g})_{I^o},$$

and

$$\delta_{\mathcal{W},I^o}\mid(\mathcal{W}\boldsymbol{g})_{I^o,\iota}=0.$$

3.3 Connections and the Chern-Weil homomorphism of Lie algebroids

Definition 13. *By a connection in a (transitive) Lie algebroid A we mean a splitting $\nabla : TM \longrightarrow A$ of the Atiyah sequence,*

$$0 \longrightarrow \boldsymbol{g} \longrightarrow A \underset{\overleftarrow{\nabla}}{\longrightarrow} TM \to 0.$$

If $\mathbf{A} = A(P)$ is the Lie algebroid of a G-principal fibre bundle $P(M, G)$ then connections in $\mathbf{A}(P)$ correspond 1-1 to usual connections in P.

Fix an arbitrary connection ∇ in A and consider:

a) the *connection form* $\omega : A \longrightarrow \boldsymbol{g}$, i.e. the 1-form on A with values in $\boldsymbol{g}$ ($\omega|\boldsymbol{g} = Id$ and $\ker \omega = \operatorname{Im} \nabla$),

$$\omega \in \Omega^1(A; \boldsymbol{g}),$$

b) the *curvature form* of ∇,

$$\Omega \in \Omega^2(A; \boldsymbol{g}),$$

defined by

$$\Omega(\xi_1, \xi_2) = \omega[\![H\xi_1, H\xi_2]\!], \quad \xi_1, \xi_2 \in \operatorname{Sec} A,$$

where $H = Id - \omega : A \longrightarrow A$ is the horizontal projection,

c) the identification $\Omega(A) = \Omega\left(M; \bigwedge \boldsymbol{g}^*\right)$.

For each point $x \in M$ the mappings $\omega_{|x} : A_{|x} \longrightarrow \boldsymbol{g}_{|x}$ and $\Omega_{|x} : \bigwedge^2 A_{|x} \longrightarrow \boldsymbol{g}_{|x}$ determine linear mappings

$$\chi_{\omega,x} : \boldsymbol{g}^*_{|x} \longrightarrow A^*_{|x} \subset \bigwedge A^*_{|x}, \quad h^* \longmapsto h^* \circ \omega_{|x},$$

and

$$\chi_{\Omega,x} : \boldsymbol{g}^*_{|x} \longrightarrow \bigwedge^2 A^*_{|x} \subset \bigwedge A^*_{|x}, \quad h^* \longmapsto h^* \circ \Omega_{|x}.$$

By the universal properties of the exterior algebra $\bigwedge \boldsymbol{g}^*_{|x}$ and the symmetric algebra $\bigvee \boldsymbol{g}^*_{|x}$ we obtain the existence and uniqueness of homomorphisms of algebras of degree 0, extending the above ones,

$$\chi^{\wedge}_{\omega,x} : \bigwedge \boldsymbol{g}^*_{|x} \longrightarrow \bigwedge A^*_{|x},$$
$$\chi^{\vee}_{\Omega,x} : \bigvee \boldsymbol{g}^*_{|x} \longrightarrow \bigwedge^{\mathrm{ev}} A^*_{|x}$$

(such that $1 \longmapsto 1$). The above morphisms define a homomorphism of algebras

$$\chi_{W,x} : W\boldsymbol{g}_{|x} = \bigwedge \boldsymbol{g}^*_{|x} \bigotimes \bigvee \boldsymbol{g}^*_{|x} \longrightarrow \bigwedge A^*_{|x},$$

$$\chi_{W,x}\left(\Phi_x \otimes \Gamma_x\right) = \chi^{\wedge}_{\omega,x}\left(\Phi_x\right) \wedge \chi^{\vee}_{\Omega,x}\left(\Gamma_x\right).$$

Passing to smooth cross-sections we obtain homomorphisms of algebras

$$\chi^{\wedge}_{\omega} : \operatorname{Sec} \bigwedge \boldsymbol{g}^* \longrightarrow \Omega\left(A\right),$$

$$\chi^{\vee}_{\Omega} : \bigoplus^{l} \operatorname{Sec} \bigvee^{l} \boldsymbol{g}^* \longrightarrow \Omega^{\mathrm{ev}}\left(A\right),$$

and

$$\chi_W : \mathcal{W}\boldsymbol{g} \to \Omega\left(A\right)$$

$$\chi_W\left(\Phi \otimes \Gamma\right) = \chi^{\wedge}_{\omega}\left(\Phi\right) \wedge \chi^{\vee}_{\Omega}\left(\Gamma\right).$$

Following [G-H-V, Vol. III, p. 341], χ_W is called the *classifying homomorphism* corresponding to the connection ∇.

One can prove that for $\Gamma \in \operatorname{Sec} \bigvee^{l} \boldsymbol{g}^*$,

$$\chi^{\vee}_{\Omega}\left(\Gamma\right) = \frac{1}{k!}\langle \Gamma, \underbrace{\Omega \vee \cdots \vee \Omega}_{l \text{ times}} \rangle$$

(the notation $\Omega \vee \cdots \vee \Omega$ comes from [G-H-V, Vol. II], it is the usual skew multiplication of differential forms whose values are multiplied according to the multilinear symmetric mapping $\vee : \boldsymbol{g} \times \cdots \times \boldsymbol{g} \longrightarrow \bigvee^{l} \boldsymbol{g}$).

Theorem 14. *(a) The classifying homomorphism χ_W commutes with the substitution operators ι_ν, $\nu \in \operatorname{Sec} \boldsymbol{g}$:*

$$\iota_\nu\left(\chi_W \Theta\right) = \chi_W\left(\iota_\nu \Theta\right).$$

(b) The homomorphism $\chi_{W,I^o} : \left(\mathcal{W}\boldsymbol{g}\right)_{I^o} \longrightarrow \boldsymbol{\Omega}\left(A\right)$, the restriction of χ_W to the invariant elements, commutes with the differentials $\delta_{\mathcal{W},I^o}$ and d_A:

$$d_A\left(\chi_{W,I^o}\Theta\right) = \chi_{W,I^o}\left(\delta_{\mathcal{W},I^o}\Theta\right).$$

As a simple consequence we obtain the Chern-Weil homomorphism of the Lie algebroid A. Consider the restriction $\chi_{W,I^o,\iota}$ of $\chi_W : \mathcal{W}\boldsymbol{g} \longrightarrow \Omega\left(A\right)$ to the horizontal invariant elements. Since

$\delta_{\mathcal{W},I^o} \mid (\mathcal{W}\boldsymbol{g})_{I^o,\iota} = 0$ we see that all differential forms in $\operatorname{Im}\chi_{W,I^o,\iota}$ are closed and horizontal:

$$\chi_{W,I^o,\iota} : (\mathcal{W}\boldsymbol{g})_{I^o,\iota} \to Z_\iota(A);$$

on the other hand, $(\mathcal{W}\boldsymbol{g})_\iota = \bigoplus_l \operatorname{Sec} \bigvee^l \boldsymbol{g}^*$, therefore

$$(\mathcal{W}\boldsymbol{g})_{I^o,\iota} = \bigoplus_l \left(\operatorname{Sec} \bigvee^l \boldsymbol{g}^*\right)_{I^o}$$

and $\Omega(M) \overset{f}{\cong} \boldsymbol{\Omega}_\iota(A)$ (via the anchor $f(\Psi)_x(v_1,...,v_k) = (\Psi)_x(\#v_1,...,\#v_k)$) and

$$\begin{array}{ccccc} \chi_{W,I^o,\iota} : (\mathcal{W}\boldsymbol{g})_{I^o,\iota} & \longrightarrow & Z_\iota(A) & & \\ \| & & \| & & \\ h_A : \bigoplus_l \left(\operatorname{Sec} \bigvee^l \boldsymbol{g}^*\right)_{I^o} & \longrightarrow & Z(M) & \longrightarrow & \mathbf{H}(M) \end{array}$$

3.4 Koszul complexes and Chevalley's theorem in the framework of Lie algebroids

We now apply the technique of Koszul complexes and Chevalley's theorem [G-H-V, Vol. III] to Lie algebroids. We recall that the adjoint representation ad_A of A on $(W\boldsymbol{g})^{k,2l} = \bigwedge^k \boldsymbol{g}^* \bigotimes \bigvee^l \boldsymbol{g}^*$ determines at each point x the adjoint representation $\theta_x^{k,2l} : \boldsymbol{g}_{|x} \longrightarrow \operatorname{End}\left(W\boldsymbol{g}_{|x}\right)^{k,2l}$, which together determine the representation on the Weil algebra $\theta_x : \boldsymbol{g}_{|x} \longrightarrow \operatorname{End}\left(W\boldsymbol{g}_{|x}\right)$. Denote by $\left(\tilde{W}\boldsymbol{g}_{|x}\right)_{I_{\theta_x}}$ the subspace of $\left(W\boldsymbol{g}_{|x}\right)_{I_{\theta_x}}$ consisting of all vectors whose homogeneous parts can be extended to globally defined cross-sections of $(W\boldsymbol{g})^{k,2l} = \bigwedge^k \boldsymbol{g}^* \bigotimes \bigvee^l \boldsymbol{g}^*$ invariant with respect to the adjoint representation of the Lie algebroid A,

$$(\mathcal{W}\boldsymbol{g})_{I^o} \cong \left(\tilde{W}\boldsymbol{g}_{|x}\right)_{I_{\theta_x}} \subset \left(W\boldsymbol{g}_{|x}\right)_{I_{\theta_x}}.$$

We assume the following (rather strong) assumptions:

(A1) the isotropy Lie algebras $\boldsymbol{g}_{|x}$ are reductive, (A2) each homogeneous invariant element

$$\Theta_x \in \left(W\boldsymbol{g}_{|x}\right)^{k,2l}_{I_{\theta_x}} = \left(\bigwedge^k \boldsymbol{g}^*_{|x} \bigotimes \bigvee^l \boldsymbol{g}^*_{|x}\right)_{I_{\theta_x}}$$

can be extended to a globally defined invariant cross-section of the vector bundle $\bigwedge^k \boldsymbol{g}^* \bigotimes \bigvee^l \boldsymbol{g}^*$, i.e.

$$(\mathcal{W}\boldsymbol{g})_{I^\circ} \cong \left(\tilde{W}\boldsymbol{g}_{|x}\right)_{I_{\theta_x}} = \left(W\boldsymbol{g}_{|x}\right)_{I_{\theta_x}}.$$

(In particular, the cohomology vector bundle $\mathbf{H}(\boldsymbol{g})$, $\mathbf{H}(\boldsymbol{g})_x = \mathbf{H}\left(\boldsymbol{g}_{|x}\right)$, is trivial).

Now we return to an arbitrarily chosen connection ∇ in the Lie algebroid A, $0 \longrightarrow \boldsymbol{g} \longrightarrow A \underset{\nabla}{\overset{\longrightarrow}{\longleftarrow}} TM \longrightarrow 0$, and take $\chi_W : \mathcal{W}\boldsymbol{g} \longrightarrow \Omega(A)$, the classifying homomorphism corresponding to the connection ∇, and its restriction to the invariant elements,

$$\chi_{W,I^\circ} : (\mathcal{W}\boldsymbol{g})_{I^\circ} \cong \left(\tilde{W}\boldsymbol{g}_{|x}\right)_{I_{\theta_x}} = \left(W\boldsymbol{g}_{|x}\right)_{I_{\theta_x}} \longrightarrow \Omega(A).$$

Now we use the assumed reductivity of the isotropy Lie algebras $\boldsymbol{g}_{|x}$. Let

$$P_x \subset \left(\bigwedge \boldsymbol{g}^*_{|x}\right)_{I_{\theta_x}}$$

be the graded primitive subspace. We recall that homogeneous primitive elements have odd degree (which implies that $\Phi \wedge \Phi = 0$ when $\Phi \in P_x$), therefore the inclusion $P_x \subset \left(\bigwedge \boldsymbol{g}^*_{|x}\right)_{I_{\theta_x}}$ extends to a homomorphism of algebras

$$\varkappa_x : \bigwedge P_x \longrightarrow \left(\bigwedge \boldsymbol{g}^*_{|x}\right)_{I_{\theta_x}}.$$

The Hopf-Samelson theorem [G-H-V, Vol. III, 5.18, Theorem III] says that if $\boldsymbol{g}_{|x}$ is reductive then $\varkappa_x$ is an isomorphism of graded algebras.

Further

$$\tau_x : P_x \longrightarrow \left(\bigvee^+ \boldsymbol{g}^*_{|x}\right)_{I_{\theta_x}}$$

denotes a fixed transgression in $\left(W\boldsymbol{g}_{|x}\right)_{I_{\theta_x}}$, i.e. a linear mapping such that

(1) τ_x is homogeneous of degree $+1$, $\tau_x : P_x^{2r-1} \longrightarrow \left(\bigvee \boldsymbol{g}_{|x}^*\right)_{I_{\theta_x}}^{2r} = \left(\bigvee^r \boldsymbol{g}_{|x}^*\right)_{I_{\theta_x}}$,

(2) for each $\Phi \in P_x$ there exists $\Omega \in W^+\left(\boldsymbol{g}_{|x}\right)_{I_{\theta_x}}$ such that

$$\delta_{W_x}\Omega = 1 \otimes \tau_x \Phi \quad \text{and} \quad \Omega - \Phi \otimes 1 \in \left(\bigwedge \boldsymbol{g}_{|x}^* \otimes \bigvee^{j\geq 1} \boldsymbol{g}_{|x}^*\right)_{I_{\theta_x}}.$$

It turns out that we can demand that Ω depends linearly on Φ, and Φ and Ω are of the same degree, i.e. that there exists a linear mapping

$$\alpha_x : P_x \longrightarrow W^+\left(\boldsymbol{g}_{|x}\right)_{I_{\theta_x}},$$

homogeneous of degree 0, such that

(*) $\delta_{W_x}\left(\alpha_x \Phi\right) = 1 \otimes \tau_x\left(\Phi\right)$,

(**) $\alpha_x\left(\Phi\right) - \Phi \otimes 1 \in \left(\bigwedge \boldsymbol{g}_{|x}^* \otimes \bigvee^{j\geq 1} \boldsymbol{g}_{|x}^*\right)_{I_{\theta_x}}$.

In the following we fix such a mapping α_x. Now we can define a Koszul complex for the Lie algebroid. To this end we recall the homomorphism

$$\chi_{W,I^o,\iota} := \left(\bigvee \boldsymbol{g}_{|x}^*\right)_{I_{\theta_x}} \cong \bigoplus_l \left(\mathrm{Sec} \bigvee^l \boldsymbol{g}^*\right)_{I^o} \longrightarrow Z_\iota\left(A\right) \cong Z\left(M\right)$$

$(Z\left(M\right) =$ closed differential forms on $M)$,

(after passing to cohomology, this yields the Chern-Weil homomorphism of A). Composing it with the transgression $\tau_x : P_x \longrightarrow \left(\bigvee^+ \boldsymbol{g}_{|x}^*\right)_{I_{\theta_x}}$ we obtain

$$\tau_A : P_x \longrightarrow \left(\bigvee \boldsymbol{g}_{|x}^*\right)_{I_{\theta_x}} \longrightarrow Z\left(M\right) \subset \Omega\left(M\right).$$

Definition 15. *In the skew tensor product of the graded algebras*

$$\Omega\left(M\right) \otimes \left(\bigwedge \boldsymbol{g}_{|x}^*\right)_{I_{\theta_x}} = \Omega\left(M\right) \otimes \bigwedge P_x$$

we introduce the operator

$$\nabla_A : \Omega\left(M\right) \otimes \bigwedge P_x \to \Omega\left(M\right) \otimes \bigwedge P_x$$

uniquely determined by the conditions:

(1) $\nabla_A(z\otimes 1) = d(z)\otimes 1,$ */d the de Rham differential*

(2) $\nabla_A(z\otimes(\Phi_0\wedge\ldots\wedge\Phi_p)) = dz\otimes(\Phi_0\wedge\ldots\wedge\Phi_p) + (-1)^q\sum_{i=0}^{p}(-1)^i\tau_A(\Phi_i)\wedge z\otimes\left(\Phi_0\wedge\ldots\widehat{i}\ldots\wedge\Phi_p\right),\quad z\in\Omega^q(M),$ $\Phi_i\in P_x$. *In particular* $\nabla_A(z\otimes\Phi) = dz\otimes\Phi + (-1)^q\tau_A(\Phi)\wedge z\otimes 1$ *and* $\nabla_A(1\otimes\Phi) = \tau_A(\Phi)\otimes 1$.

Lemma 16. *The operator* ∇_A *is an antiderivation of square* 0*, homogeneous of degree* $+1$.

Definition 17. *The pair* $\left(\Omega(M)\otimes\bigwedge P_x,\nabla_A\right)$ *is called the Koszul complex of the Lie algebroid* A.

We see that the Koszul complex for a Lie algebroid depends only on the base manifold and the Chern-Weil homomorphism of A.

Now we define a Chevalley homomorphism. Take the restriction of the classifying homomorphism $\chi_W:\mathcal{W}\boldsymbol{g}\longrightarrow\Omega(A)$ to the invariant tensors,

$$\chi_{W,I^o}:(\mathcal{W}\boldsymbol{g})_{I^o}\cong\left(\tilde{W}\boldsymbol{g}_{|x}\right)_{I_{\theta_x}} = \left(W\boldsymbol{g}_{|x}\right)_{I_{\theta_x}}\longrightarrow\Omega(A).$$

Composing it with the mapping $\alpha_x:P_x\to W^+\left(\boldsymbol{g}_{|x}\right)_{I_{\theta_x}}\subset\left(W\boldsymbol{g}_{|x}\right)_{I_{\theta_x}}$,

$$P_x\xrightarrow{\alpha_x}W^+\left(\boldsymbol{g}_{|x}\right)_{I_{\theta_x}}\subset\left(W\boldsymbol{g}_{|x}\right)_{I_{\theta_x}}\cong(\mathcal{W}\boldsymbol{g})_{I^o}\xrightarrow{\chi_{W,I^o}}\Omega(A,)$$

we obtain a linear mapping homogeneous of degree 0,

$$\vartheta_A:P_x\longrightarrow\Omega(A).$$

Hence, since $\Omega(A)$ is anticommutative and $P_x^k=0$ for even k, ϑ_A extends to a homomorphism of graded algebras

$$\vartheta_A^\wedge:\bigwedge P_x\longrightarrow\Omega(A).$$

Finally, we extend $\vartheta_A^\wedge:\bigwedge P_x\longrightarrow\Omega(A)$ to a homomorphism of graded algebras

$$\vartheta_A:\Omega(M)\otimes\bigwedge P_x\longrightarrow\Omega(A)$$

by setting

$$\vartheta_A(z\otimes\Phi) = \#_A^*(z)\wedge\vartheta_A^\wedge(\Phi)$$

($\#_A^*(z)$ is the pull back, via the anchor $\#_A$, of the differential form $z\in\Omega(M)$ to a horizontal one on the Lie algebroid A).

Definition 18. *The homomorphism $\vartheta_A : \Omega(M) \otimes \bigwedge P_x \longrightarrow \Omega(A)$ is called the Chevalley homomorphism of A associated with the connection ∇ and the linear map α_x.*

Theorem 19 (The fundamental theorem)**.** *(A) The Chevalley homomorphism ϑ_A is a homomorphism of graded differential algebras*

$$\vartheta_A : \left(\Omega(M) \otimes \bigwedge P_x, \nabla_A\right) \longrightarrow (\Omega(A), d_A).$$

(B) Under the assumptions (A1) and (A2), i.e. that the isotropy Lie algebras $\boldsymbol{g}_{|x}$ are reductive, and $\left(\tilde{W}\boldsymbol{g}_{|x}\right)_{I_{\theta_x}} = \left(W\boldsymbol{g}_{|x}\right)_{I_{\theta_x}}$, the induced homomorphism in cohomology

$$\vartheta_A^{\#} : \mathbf{H}\left(\Omega(M) \otimes \bigwedge P_x, \nabla_A\right) \longrightarrow \mathbf{H}(A)$$

is an isomorphism of graded algebras.

Proof. (A) It is sufficient to check the equality $d_A \circ \vartheta_A = \vartheta_A \circ \nabla_A$ on simple tensors $z \otimes 1$ and $1 \otimes \Phi$ ($\Phi \in P_x$) only. We have

$$d_A \circ \vartheta_A (z \otimes 1) = d_A (\#_A^* z) = \#_A^* (dz) = \vartheta_A (dz \otimes 1) = \vartheta_A \circ \nabla_A (z \otimes 1),$$

and

$$\begin{aligned} d_A \circ \vartheta_A (1 \otimes \Phi) &= d_A \left(\vartheta_A^{\wedge} (\Phi)\right) = d_A \left(\chi_{W,I^o} (\alpha_x (\Phi))\right) \\ &\overset{\text{Th (14)}}{=} \chi_{W,I^o} \left(\delta_{\mathcal{W},I^o} (\alpha_x (\Phi))\right) = \chi_{W,I^o} \left(\delta_{W_x} (\alpha_x (\Phi))\right) \\ &\overset{(*)}{=} \chi_{W,I^o} (1 \otimes \tau_x (\Phi)) = \chi_{W,I^o} (\tau_x (\Phi)) = \#_A^* (\tau_A \Phi) \\ &= \vartheta_A (\tau_A \Phi \otimes 1) = \vartheta_A \circ \nabla_A (1 \otimes \Phi). \end{aligned}$$

(B) The proof is analogous to that in the classical case for principal fibre bundles [G-H-V, Vol. III, 9.3-4, p. 359]: we use some spectral sequences and the comparison theorem for the first terms (the mapping induced on the first terms is an isomorphism).

Step 1. Filtrations: For a given Lie algebroid A with the Atiyah sequence $0 \longrightarrow \boldsymbol{g} \longrightarrow A \longrightarrow TM \longrightarrow 0$ we consider the pair of real (infinite dimensional) Lie algebras $(\Gamma(A), \Gamma(\boldsymbol{g}))$ of global cross-sections of A and $\boldsymbol{g}$.

Following [H-S], [K-M-2], we introduce the Hochschild-Serre filtration in $\Omega(A)_j$ in $\Omega(A)$ as follows:

$$\Omega(A)_j = \left\{ \begin{bmatrix} \Omega(A) & \text{for} & j \leq 0, \\ \bigoplus_{k\geq j} \ \Omega(A)_j^k & \text{for} & j > 0. \end{bmatrix} \right.$$

where $\Omega(A)_j^k$ consists of all those k-differential forms $z \in \Omega^k(A)$ for which

$$z(\xi_1, ..., \xi_k) = 0$$

whenever $k-j+1$ of the arguments $\xi_i \in \Gamma(A)$ belong to $\Gamma(\boldsymbol{g})$. In this way we obtain a graded filtered differential space and its spectral sequence $\left(E_{A,s}^{j,i}, d_{A,s}\right)$.

Analogously, following [G-H-V, Vol. III] we introduce in the space $\Omega(M) \otimes \bigwedge P_x$ the filtration

$$\left(\Omega(M) \otimes \bigwedge P_x\right)_j = \bigoplus_{k\geq j} \ \Omega(M)^k \otimes \bigwedge P_x.$$

We obtain a graded filtered differential space and its spectral sequence $\left(E_s^{j,i}, d_s\right)$.

Step 2. We show that the Chevalley homomorphism ϑ_A is filtration preserving. Firstly we notice that $\vartheta_A(z \otimes 1) = \#_A^*(z)$ and $\vartheta_A(1 \otimes \Phi) - \chi_{W,I^o}(\Phi \otimes 1) \in \Omega(A)_1$. The first statement is obvious. To prove the second, it is sufficient to consider the case $\Phi \in P_x$. According to (**) above it follows that

$$\begin{aligned} \vartheta_A(1 \otimes \Phi) - \chi_{W,I^o}(\Phi \otimes 1) &= \chi_{W,I^o}(\alpha_x \Phi) - \chi_{W,I^o}(\Phi \otimes 1) \\ &= \chi_{W,I^o}(\alpha_x \Phi - \Phi \otimes 1) \in \Omega(A)_1. \end{aligned} \tag{1}$$

By definition, $\vartheta_A\left[\Omega(M)^k \otimes 1\right] \subset \Omega(A)_k$. Since $\left(\Omega(M) \otimes \bigwedge P_x\right)_j$ is the ideal generated by $\bigoplus_{k\geq j} \ \Omega(M)^k \otimes 1$, and since $\Omega(A)_j$ is an ideal, this implies that ϑ_A preserves filtrations.

Step 3. We show that the mapping of the first terms of the spectral sequences,

$$\vartheta_{A,1} : E_1 \longrightarrow E_{A,1},$$

is an isomorphism. In view of the Comparison Theorem the induced homomorphism in cohomology $\vartheta_A^\# : \mathbf{H}\left(\Omega(M) \otimes \bigwedge P_x, \nabla_A\right) \longrightarrow \mathbf{H}(A)$ is an isomorphism.

We start by calculating the differential operators d_0 in E_0 and $d_{A,0}$ in $E_{A,0}$. It is immediate from the definitions that

$$\nabla_A : \Omega(M)^k \otimes \bigwedge^l P_x \longrightarrow \left(\Omega(M) \otimes \bigwedge P_x\right)_{k+1}, \quad k, l \geq 0.$$

It follows that $d_0 = 0$. On the other hand, recall from [K-M-2, Conclusion 5.2] that $E^j_{A,0} = \Omega^j\left(M; \bigwedge \boldsymbol{g}^*\right)$ and that the differential $d_{A,0}$ becomes the Chevalley-Eilenberg differential of values at each point.

Now, we show that $\vartheta_{A,0} : E_0 \longrightarrow E_{A,0}$ simply comes from the inclusion map

$$j : \Omega(M) \otimes \bigwedge P_x = \Omega(M) \otimes \left(\bigwedge \boldsymbol{g}^*_{|x}\right)_{I_{\theta_x}} \longrightarrow \Omega\left(M; \bigwedge \boldsymbol{g}^*\right)$$

and its values are $d_{A,0}$-closed. In fact, j is homogeneous of bidegree zero. Thus we need only show that

$$\vartheta_A - j : \Omega^k(M) \otimes \bigwedge P_x \longrightarrow \Omega(A)_{k+1}.$$

But $j(z \otimes \Phi) = \#^*_A \wedge \chi_{W,I^o}(\Phi \otimes 1)$, and so property (1) yields, for $z \in \Omega^k(M)$,

$$(\vartheta_A - j)(z \otimes \Phi) = \#^*_A z \wedge (\vartheta_A(1 \otimes \Phi) - \chi_{W,I^o}(\Phi \otimes 1)) \in \Omega(A)_{k+1}.$$

To prove Step 3 we need only show that $(\vartheta_{A,0})^\# : \mathbf{H}(E_0, d_0) \longrightarrow \mathbf{H}(E_{A,0}, d_{A,0})$ is an isomorphism. In view of the formulae for d_0 and $d_{A,0}$ it remains to show that the inclusion map j induces an isomorphism

$$j^\# : \Omega(M) \otimes \bigwedge P_x = \Omega(M) \otimes \left(\bigwedge \boldsymbol{g}^*_{|x}\right)_{I_{\theta_x}} \longrightarrow \left(\Omega\left(M; \bigwedge \boldsymbol{g}^*\right), d_{A,0}\right).$$

Since the Lie algebras $\boldsymbol{g}_{|x}$ are reductive (assumption (A1)), by the structural theorem for reductive Lie algebras [G-H-V, Vol. III, s. 5.12, Theorem 1] we have $\left(\bigwedge \boldsymbol{g}^*_{|x}\right)_{I_{\theta_x}} = \mathbf{H}\left(\boldsymbol{g}_{|x}\right)$. Therefore, the isomorphism property of $j^\#$ follows immediately from assumption (A2): $\left(\Omega\left(M; \bigwedge \boldsymbol{g}^*\right), d_{A,0}\right) = \Omega(M; \mathbf{H}(\boldsymbol{g})) = \Omega\left(M; \mathbf{H}\left(\boldsymbol{g}_{|x}\right)\right)$. The proof of the fundamental theorem is now complete. $\square$

Problem 20. *What can we do in the case when* $\left(\tilde{W}\boldsymbol{g}_{|x}\right)_{I_{\theta_x}} \subsetneq \left(W\boldsymbol{g}_{|x}\right)_{I_{\theta_x}}$ *to calculate* $\mathbf{H}(A)$ *? [The simplest examples of this case come from connected pfb's with disconnected structural Lie groups].*

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The Lefschetz Theorem for multivalued maps

A subcollection of multivalued maps called s-maps is introduced. Then to a self s-map f of a finite connected CW-complex an integer $\mathcal{L}_s(f)$ is associated and an analog of the Lefschetz Fixed Point Theorem is proved.

1 Introduction

The Lefschetz Fixed Point Theorem states that if X is a sufficiently nice space and $f\colon X \to X$ is a singlevalued continuous map, then it is possible to associate to f an integer $\mathcal{L}(f)$ such that if $\mathcal{L}(f) \neq 0$, then f has a fixed point. The number $\mathcal{L}(f)$ is called the Lefschetz number of f and is a well known and very useful homotopical invariant.

In the literature one can find many tries of generalizations of the Lefschetz number to the case of multivalued maps. The paper [2] presents the Lefschetz number defined for maps for which the image of any two different points has the same finite number of elements. More general approches regarding to acyclic and admissible maps are presented in [3]. Authors of [6] consider multivalued maps with an additional algebraic structure called the weight of a map. They construct for such maps the Lefschetz number using the Darbo homology functor, but that number essentially depends on the weight as well.

The main goal of this paper is to introduce a subcollection of multivalued maps, called s-maps for which it is possibile to define the Lefschetz number in a new way. In Section 2 we recall some basic information about the ordinary Lefschetz number and the Lefschetz set of admissible maps presented in [3]. Then in Section 3 we show a categorical construction which leads to an extension of some definitions to larger categories. We apply such a construction to define a subcollection of multivalued maps called s-maps and the Lefschetz number of them as well. Moreover, we prove an analog of the Lefschetz Fixed Point Theorem for s-maps (Theorem 3.9). Next in Section 4 we compare our approach with the Lefschetz set of admissible maps. We present some examples of s-maps which are not admissible. At the end of the section we use the categorical construction again to define s-admissible maps which generalize both admissible and s-maps. Next we formulate the Lefschetz Fixed Point Theorem for such maps (Theorem 4.9).

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2 Preliminaries

First we recall some basic information about multivalued maps. More details one can find in [3].

Let X, Y be two topological spaces and assume that for every point $x \in X$ a nonempty compact subset $\varphi(x) \subseteq Y$ is given. In this case, we say that φ is a *multivalued map* from X to Y and we write $\varphi\colon X \multimap Y$.

Let $\varphi\colon X \multimap Y$ be a multivalued map and $A \subseteq X$, then the *image* of A under φ is the set

$$\varphi(A) = \bigcup_{x \in A} \varphi(x).$$

Let $\varphi\colon X \multimap Y$ be a multivalued map and $B \subseteq Y$, then the *large preimage* of B under φ is the set

$$\varphi^{-1}(B) = \{x \in X \mid \varphi(x) \cap B \neq \emptyset\}.$$

If $\varphi\colon X \multimap Y$ and $\psi\colon Y \multimap Z$ are two multivalued maps, then for any $C \subseteq Z$ we have

$$(\psi \circ \varphi)^{-1}(C) = \varphi^{-1}(\psi^{-1}(C)).$$

A multivalued map $\varphi\colon X \multimap Y$ is called *upper semicontinuous* (*u.s.c.*), provided for every closed $B \subseteq Y$ the set $\varphi^{-1}(B)$ is closed in X. If $\varphi\colon X \multimap$

Y and $\psi\colon Y \multimap Z$ are u.s.c. maps, then the composition $\psi \circ \varphi\colon X \multimap Z$ is also u.s.c..

Remark 2.1. Let $f\colon X \to Y$ be a singlevalued continuous map onto Y. Then its inverse can be considered as a multivalued map $f^{inv}\colon Y \multimap X$ defined by $f^{inv}(y) = f^{-1}(y)$ for $y \in Y$. If f is closed, then f^{inv} is u.s.c.. Later we write f^{-1} instead of f^{inv}.

Now we recall the Lefschetz Fixed Point Theorem for singlevalued maps. For details check [1], [4] and [5]. Denote by $\mathcal{C}$ the collection of all finite connected CW-complexes. Let $X \in \mathcal{C}$ and $f\colon X \to X$ be a singlevalued continuous map. Recall that we have a well defined integer $\mathcal{L}(f) = \sum_{k\in\mathbb{Z}} (-1)^k \mathrm{tr}(f_k)$, called the *Lefschetz number* of f, where $f_k\colon H_k(X,\mathbb{Q}) \to H_k(X,\mathbb{Q})$ are maps induced by f on the rational homology groups and $\mathrm{tr}(f_k)$ denotes the trace of the homomorphism f_k.

Remark 2.2. When we work in $\mathcal{C}$ there is no difference which homology functor we use, but if X is an arbitrary topological space, then by $H_k(X,\mathbb{Q})$ we mean the k-th Čech rational homology group of X.

Notice that the Lefschetz number has the following very useful properties, which are a simple consequences of analogous properties of the trace:

Proposition 2.3. *Let X, $Y \in \mathcal{C}$, $f\colon X \to Y$ and $g\colon Y \to X$ be two singlevalued continuous maps, then $\mathcal{L}(fg) = \mathcal{L}(gf)$.*

Corollary 2.4. *Let X, $Y \in \mathcal{C}$, $g\colon X \to Y$ be a homeomorphism and $f\colon X \to X$ be a singlevalued continuous map, then $\mathcal{L}(f) = \mathcal{L}(gfg^{-1})$.*

There is also a property which connects the Lefschetz number of a map with the Lefschetz number of its prime iteration:

Theorem 2.5 (The mod p Theorem [4])**.** *Let $X \in \mathcal{C}$, $f\colon X \to X$ be a contiuous map and p be a prime. Then $\mathcal{L}(f^p) \equiv \mathcal{L}(f) \bmod p$.*

The most famouse and important application of the Lefschetz number is:

Theorem 2.6 (Lefschetz Fixed Point Theorem [5])**.** *Let $X \in \mathcal{C}$ and $f\colon X \to X$ be a singlevalued continuous map. If $\mathcal{L}(f) \neq 0$, then f has a fixed point.*

To recall the construction of the Lefschetz number of multivalued admissible maps, we need some definitions. All results presented below are stated in [3].

A space X is called *acyclic*, when:
(i) $H_k(X, \mathbb{Q}) = 0$ for all $k \geq 1$;
(ii) $H_0(X, \mathbb{Q}) = \mathbb{Q}$.

A singlevalued continuous map $f\colon X \to Y$ is called *proper*, provided for every compact $K \subseteq Y$ the set $f^{-1}(K)$ is compact.

A singlevalued continuous map $f\colon X \to Y$ is called a *Vietoris map*, provided the following conditions hold:
(i) $f\colon X \to Y$ is proper;
(ii) the set $f^{-1}(y)$ is acyclic for all $y \in f(X)$.

Vietoris maps have the following important property:

Theorem 2.7 (Vietoris [3])**.** *If X, $Y \in \mathcal{C}$ and $f\colon X \to Y$ is a Vietoris map, then the induced homomorphism $f_k\colon H_k(X, \mathbb{Q}) \to H_k(Y, \mathbb{Q})$ is an isomorphism for all $k \geq 0$.*

A multivalued map $\varphi\colon X \multimap Y$ is called *admissible*, provided there exist a space Z and two continuous maps $p\colon Z \to X$ and $q\colon Z \to Y$ such that:
(i) p is a Vietoris map;
(ii) $q(p^{-1}(x)) \subseteq \varphi(x)$ for all $x \in X$.
We write $(p, q) \subseteq \varphi$ when maps p and q are as above.

The *Lefschetz set* of an admissible map $\varphi\colon X \multimap X$ is defined by:

$$\mathcal{L}_a(\varphi) = \left\{ \sum_{k \in \mathbb{Z}} (-1)^k \operatorname{tr}(q_k p_k^{-1}) \mid (p, q) \subseteq \varphi \right\}.$$

Remark 2.8. Let $X, Y \in \mathcal{C}$ and $\varphi\colon X \multimap Y$ be an admissible map. It is easy to see, that if $(p, q) \subseteq \varphi$, then $qp^{-1}\colon X \multimap Y$ is u.s.c. and the set $qp^{-1}(x)$ is connected for all $x \in X$.

The most important application of the Lefschetz set is:

Theorem 2.9 (Lefschetz Fixed Point Theorem for admissible maps [3])**.** *Let $X \in \mathcal{C}$ and $\varphi\colon X \multimap X$ be an admissible map. If $\mathcal{L}_a(\varphi) \neq \{0\}$, then φ has a fixed point.*

Remark 2.10. In [3] a collection of spaces for which it is possible to consider admissible maps is larger than $\mathcal{C}$. Moreover, in [7] there is considered a broader class of maps for which it is possible to define the Lefschetz set.

3 The Lefschetz number of s-maps

In this section we introduce a subcollection of multivalued maps called s-maps and investigate their properties. First, we show a very usefull categorical construction which helps us in defining s-maps.

Definition 3.1. Let $\mathcal{D}$ be a category and $\mathcal{C}$ its subcategory, not necessary full. Define a category $(\mathcal{D}, \mathcal{C})$ as follows:
(i) object of $(\mathcal{D}, \mathcal{C})$ are quadruples (X, A, r, s), where $X \in \mathcal{D}$, $A \in \mathcal{C}$, $r\colon X \to A$ and $s\colon A \to X$ are morphisms in $\mathcal{D}$ such that $rs = \mathrm{id}_A$;
(ii) $\mathrm{Mor}_{(\mathcal{D},\mathcal{C})}((X, A, r, s), (Y, B, t, q)) = \{(\varphi, f) \in \mathcal{D} \times \mathcal{C} \mid \varphi = qfr,\ f \in \mathrm{Mor}_{\mathcal{C}}(A, B)\}$;
(iii) a composition law in $(\mathcal{D}, \mathcal{C})$ is induced from the composition laws in $\mathcal{D}$ and $\mathcal{C}$;
(iv) $\mathrm{id}_{(X,A,r,s)} = (sr, \mathrm{id}_A)$.

Observe that the composition of morphisms in the category $(\mathcal{D}, \mathcal{C})$ is well defined because if $(\varphi, f) \in \mathrm{Mor}_{(\mathcal{D},\mathcal{C})}((X, A, r, s), (Y, B, t, q))$ and $(\psi, g) \in \mathrm{Mor}_{(\mathcal{D},\mathcal{C})}((Y, B, t, q), (Z, C, l, m))$, then $\psi\varphi = lgtqfr = lgfr$, because $tq = \mathrm{id}_Y$, so $(\psi\varphi, gf) \in \mathrm{Mor}_{(\mathcal{D},\mathcal{C})}((X, A, r, s), (Z, C, l, m))$.

Notice that if $\varphi = qfr$, then $f = t\varphi s$, so f is uniquely determinded by φ (when suitable r, s, t and q are choosen).

Now let $\mathcal{D}$ be a category of topological spaces and multivalued u.s.c. maps and $\mathcal{C}$ its subcategory of finite connected CW-complexes and singlevalued continuous maps. Consider the category $(\mathcal{D}, \mathcal{C})$ for such $\mathcal{D}$ and $\mathcal{C}$.

Definition 3.2. Let $X \in \mathcal{D}$ and $\varphi\colon X \multimap X$ be a multivalued u.s.c. map. The map φ is called an *s-map* if there exist:
(i) $A \in \mathcal{C}$;
(ii) a singlevalued continuous map $f_\varphi\colon A \to A$;
(iii) a singlevalued continuous surjection $r\colon X \to A$;
(iv) a multivalued u.s.c. map $s\colon A \multimap X$;
such that:
(a) $(X, A, r, s) \in (\mathcal{D}, \mathcal{C})$;
(b) $(\varphi, f_\varphi) \in \mathrm{Mor}_{(\mathcal{D},\mathcal{C})}((X, A, r, s), (X, A, r, s))$.

If $\varphi\colon X \multimap X$ is an s-map and f_φ is a map like in the above definition, then we say that the morphism (φ, f_φ) represents φ in $(\mathcal{D}, \mathcal{C})$.

If $X \in \mathcal{C}$ and $\varphi\colon X \to X$ is a singlevalued continuous map, then clearly φ is an s-map, because it is enough to take $A = X$, $f_\varphi = \varphi$ and $r = s = \mathrm{id}_X$. This factorization is called *standart*. Of course for a singlevalued continuous map there can exsist factorizations different to the standart one.

Example 3.3. A map $\varphi\colon [0,2] \to [0,2]$ given by $\varphi(x) = 0$ for all $x \in [0,2]$ is singlevalued, so we have the standard factorization. On the other hand, we can choose a different morphism in $(\mathcal{D}, \mathcal{C})$ which represet φ, for example $r\colon [0,2] \to [0,1]$ is given by

$$r(x) = \begin{cases} x & \text{for } x \in [0,1]; \\ 1 & \text{for } x \in (1,2]; \end{cases}$$

$s\colon [0,1] \multimap [0,2]$ is the inverse of r, so $s(x) = r^{-1}(x)$ for all $x \in [0,1]$ and $f_\varphi\colon [0,1] \to [0,1]$ is defined by $f_\varphi(x) = 0$ for all $x \in [0,1]$.

For selfmorphisms in category $\mathcal{C}$ we have a well defined Lefschetz number. We can extend this definition to the category $(\mathcal{D}, \mathcal{C})$ by taking $\mathcal{L}(\varphi, f_\varphi) = \mathcal{L}(f_\varphi)$. Our goal is to show that if an s-map $\varphi\colon X \multimap X$ is represented by two different morphisms (φ, f_φ) and (φ, g_φ), then $\mathcal{L}(f_\varphi) = \mathcal{L}(g_\varphi)$. To show that we need first to prove some lemmas:

Lemma 3.4. *If an s-map $\varphi\colon X \to X$ is represented by pairs (φ, f_φ) and (φ, g_φ) are such that $\varphi = sf_\varphi r$ and $\varphi = tg_\varphi q$ for suitable s, r and q, t, then:*
(i) $f_\varphi r = rtg_\varphi q$;
(ii) $qsf_\varphi = g_\varphi qs$;
(iii) $g_\varphi qsrt = g_\varphi$;
(iv) rtg_φ *is a singlevalued continuous map;*
(v) qsf_φ *is a singlevalued continuous map;*
(vi) $g_\varphi qs$ *is a singlevalued continuous map.*

Proof. (i), (ii) and (iii) are easy consequences of equalities $sf_\varphi r = tg_\varphi q$, $qt = \mathrm{id}$ and $rs = \mathrm{id}$. Let now prove (iv). Using (i) we have that $rtg_\varphi q$ is a singlevalued continuous map. Moreover, q is a singlevalued continuous surjection, so (iv) follows. Property (v) is analogous to (iv) and (vi) is a consequence of (ii) and (v). □

Lemma 3.5. *If an s-map $\varphi\colon X \to X$ is represented in $(\mathcal{D}, \mathcal{C})$ by pairs (φ, f_φ) and (φ, g_φ), then $\mathcal{L}(f_\varphi^n) = \mathcal{L}(g_\varphi^n)$ for $n \geq 2$.*

Proof. Let $\varphi = sf_\varphi r$ and $\varphi = tg_\varphi q$, then using Proposition 2.3 and Lemma 3.4 we obtain:
$\mathcal{L}(f_\varphi^n) = \mathcal{L}((rtg_\varphi qs)^n) = \mathcal{L}(rtg_\varphi^n qs) = \mathcal{L}(rtg_\varphi^{n-1} g_\varphi qs) = \mathcal{L}(g_\varphi qsrtg_\varphi^{n-1}) = \mathcal{L}(g_\varphi^n)$. □

As an easy consequence of Theorem 2.5 and Lemma 3.5 we get:

Proposition 3.6. *If an s-map $\varphi\colon X \to X$ is represented in $(\mathcal{D}, \mathcal{C})$ by pairs (φ, f_φ) and (φ, g_φ), then $\mathcal{L}(f_\varphi) = \mathcal{L}(g_\varphi)$.*

Remark 3.7. Example 3.3 shows that we cannot prove the above proposition directly as Lemma 3.5 because the composition $sr\colon X \multimap X$ does not have to be singlevalued.

Now we are in a position to define the Lefschetz number of s-maps:

Definition 3.8. Let $\varphi\colon X \multimap X$ be an s-map. The Lefschetz number of φ is a number $\mathcal{L}_s(\varphi) = \mathcal{L}(f_\varphi)$, where (φ, f_φ) represents φ in $(\mathcal{D}, \mathcal{C})$. According to Proposition 3.6 this number is well defined.

If $f\colon X \to X$ is a singlevalued continuous map then $\mathcal{L}_s(f) = \mathcal{L}(f)$. To show this it is enough to take the standart factorization.

Now we prove an analog of the Lefschetz Fixed Point Theorem for s-maps:

Theorem 3.9 (Lefschetz Fixed Point Theorem)**.** *Let $\varphi\colon X \multimap X$ be an s-map and $\mathcal{L}_s(\varphi) \neq 0$, then φ has a fixed point.*

Proof. The map φ is an s-map. Let suitable A, f_φ, r and s be choosen. Acoording to the definition $\mathcal{L}_s(\varphi) = \mathcal{L}(f_\varphi)$, where $\varphi = sf_\varphi r$. The map $f_\varphi\colon A \to A$ is singlevalued continuous and $A \in \mathcal{C}$, so the ordinary Lefschetz Fixed Point Theorem implies that f_φ has a fixed point $a = r(x)$ for some $x \in X$. Let $z \in s(r(x))$, then $\varphi(z) = \varphi(x)$, because $r(z) = r(x)$. Therefore, we have $z \in sr(x) = sf_\varphi r(x) = \varphi(x) = \varphi(z)$, so z is a fixed point of φ. $\square$

The easiest way to show that $\varphi\colon X \multimap X$ is an s-map, it is to find an equivalence relation R on X such that $A = X/R$ and $r\colon X \to A$ is the canonical projection. If $\varphi\colon X \multimap X$, then

$$R = \{(x, y) \in X \times X \mid \varphi(x) = \varphi(y)$$

and

$$\varphi^{-1}(x) = \varphi^{-1}(y) \neq \emptyset\} \cup \{(x, x) \mid x \in X\}$$

is called a *canonical relation* for φ. If we use the canonical relation, then we write X_R istead of A.

Now we present some examples of s-maps and find their Lefschetz numbers:

Example 3.10. Let S^n be the n-sphere and $\varphi\colon S^n \to S^n$ be such that $\varphi(x) = S^n$ for all $x \in S^n$. Let R be the canonical relation for φ. Then $X_R = \{*\}$, where $\{*\}$ denotes the one point space. We have $\varphi = sf_\varphi r$, where $r\colon S^n \to \{*\}$ is given by $r(x) = \{*\}$ for every $x \in S^n$, $s\colon \{*\} \multimap S^n$ is defined by $s(*) = S^n$ and $f_\varphi = \mathrm{id}_{\{*\}}$. Therefore φ is an s-map and $\mathcal{L}_s(\varphi) = \mathcal{L}(f_\varphi) = \mathcal{L}(\mathrm{id}_{\{*\}}) = 1$.

Example 3.11. Let $X \in \mathcal{C}$ and $f\colon S^1 \to S^1$ be a continuous singlevalued map. Define $\varphi\colon S^1 \times X \multimap S^1 \times X$ by $\varphi(b,x) = \{f(b)\} \times X$. Let R be the canonical relation for φ. Then $X_R = S^1$, $r\colon S^1 \times X \to S^1$ is given by $r(b,x) = b$ for all $(b,x) \in S^1 \times X$, $s\colon S^1 \multimap S^1 \times X$ is given by $s(b) = \{b\} \times X$ and $f_\varphi = f$. We have $\mathcal{L}_s(\varphi) = \mathcal{L}(f)$.

4 Comparision of the Lefschetz numbers

In this section we compare the Lefschetz number of s-maps with the Lefschetz set of admissible maps. One may expect that there is some coincidence between those two conceptions, but as we see in some examples they give different results. This leads to a definition of s-admissible maps which generalizes both admissible and s-maps. We start this section with analysing some examples.

Our first example shows a situation when $\mathcal{L}_a(\varphi) \neq \{\mathcal{L}_s(\varphi)\}$:

Example 4.1. Let $\varphi\colon S^n \multimap S^n$ be such that $\varphi(x) = S^n$ for all $x \in S^n$. We have shown in Example 3.10, that this is an s-map and $\mathcal{L}_s(\varphi) = 1$. On the other hand, the map φ is admissible and $\mathcal{L}_a(\varphi) = \mathbb{Z}$, because $(\mathrm{id}_{S^n}, f) \subset \varphi$ for all singlevalued continuous maps $f\colon S^1 \to S^1$.

In the previous example we have $\{\mathcal{L}_s(\varphi)\} \subseteq \mathcal{L}_a(\varphi)$, but that is not true in general.

Example 4.2. Let $\varphi\colon [0,2] \multimap [0,2]$ be given by:

$$\varphi(x) = \begin{cases} x & \text{for } x \in (0,2); \\ \{0,2\} & \text{for } x \in \{0,2\}. \end{cases}$$

Then φ is both admissible and an s-map. We have

$$\mathcal{L}_a(\varphi) = \left\{ \sum_{k\in\mathbb{Z}} (-1)^k \mathrm{tr}(q_k p_k^{-1}) \mid (p,q) \subseteq \varphi \right\} = \{\mathcal{L}(\mathrm{id}_{[0,2]})\} = \{1\},$$

because we cannot choose p and q such that $qp^{-1} \neq \mathrm{id}_{[0,2]}$ (see Remark 2.8). On the other hand, we have $\mathcal{L}_s(\varphi) = \mathcal{L}(f_\varphi) = \mathcal{L}(\mathrm{id}_{S^1}) = 0$, because X_R is homeomorphic to S^1 and using Proposition 2.4 we can replace in our calculations the map f_φ by id_{S^1}.

In next two examples we show s-maps which are not admissible. As a consequence for those maps only the Lefschetz number of s-maps is possible to define.

Example 4.3. Let $\varphi\colon [0,2] \multimap [0,2]$ be given by:

$$\varphi(x) = \begin{cases} x+1 & \text{for } x \in [0,1); \\ \{0,2\} & \text{for } x = 1; \\ x-1 & \text{for } x \in (1,2]. \end{cases}$$

This map is not admissible (see Remark 2.8). On the other hand, φ is an s-map and we have $\mathcal{L}_s(\varphi) = \mathcal{L}(f_\varphi) = \mathcal{L}(\mathrm{id}_{S^1}) = 0$, because X_R is homeomorphic to S^1 and using Proposition 2.4 we can think that f_φ is a rotation by an angle π which is homotopic to the identity map on S^1.

Example 4.4. Let $\varphi\colon [0,2] \to [0,2]$ be given by:

$$\varphi(x) = \begin{cases} -x+1 & \text{for } x \in [0,1); \\ \{0,2\} & \text{for } x = 1; \\ -x+3 & \text{for } x \in (1,2]. \end{cases}$$

Then φ is not admissible, but φ is an s-map and we have $\mathcal{L}_s(\varphi) = \mathcal{L}(f_\varphi) = 2$, because we can think that $f_\varphi\colon S^1 \to S^1$ and has a degree equal -1.

Remark 4.5. After studing two previous examples one can easy see that for any integer n it is possible to find a multivalued map φ which is not admissible, but is an s-map and $\mathcal{L}_s(\varphi) = n$. Namely, it is enough to take a multivalued map $\varphi\colon [0,2] \multimap [0,2]$ which is not admissible and a suitable map $f_\varphi\colon S^1 \to S^1$ has a degree $1-n$.

Remark 4.6. The maps from the last two examples can be considered as the multivalued weighted maps (check [6] for a definition), but only trivial weight is possible for those maps.

Now we formulate a definition which generalizes both admissible and s-maps. Let $\mathcal{D}$ be a category of spaces and multivalued maps and $\mathcal{C}$ be its subcategory of finite connected CW-complexes and admissible maps.

Definition 4.7. Let $X \in \mathcal{D}$ and $\varphi\colon X \multimap X$ be a multivalued map. The map φ is called an *s-admissible map* if there exist:
(i) $A \in \mathcal{C}$;
(ii) an admissible map $\beta_\varphi\colon A \multimap A$;
(iii) a singlevalued continuous surjection $r\colon X \to A$;
(iv) a multivalued u.s.c. map $s\colon A \multimap X$;
such that:
(a) $s\beta_\varphi r(x) \subseteq \varphi(x)$ for all $x \in X$;
(b) $(X, A, r, s) \in (\mathcal{D}, \mathcal{C})$.

Let $\varphi\colon X \multimap X$ be s-admissible. Denote by $(\mathcal{D}, \mathcal{C})_\varphi$ the set of all maps β_φ which are like in the above definition.

Definition 4.8. The *Lefschetz set* of s-admissible map is a set:

$$\mathcal{L}_s(\varphi) = \bigcup_{\beta_\varphi \in (\mathcal{D},\mathcal{C})_\varphi} \mathcal{L}_a(\beta_\varphi).$$

Theorem 4.9 (Lefschetz Fixed Point Theorem)**.** *Let $X \in \mathcal{C}$ and $\varphi\colon X \multimap X$ be an s-admissible map. If $\mathcal{L}_s(\varphi) \neq \{0\}$, then φ has a fixed point.*

Proof. We choose suitable s, r and β_φ such that $\mathcal{L}_a(\beta_\varphi) \neq \{0\}$. Than we use Theorem 2.9 and following the proof of Theorem 3.9 we obtain that $s\beta_\varphi r$ has a fixed point, which is also a fixed point of φ. □

Remark 4.10. The easiest way to show that $\varphi\colon X \multimap X$ is s-admissible is to find an admissible map $\psi\colon X \multimap X$ such that $\psi(x) \subseteq \varphi(x)$ for all $x \in X$ or an s-map $\eta\colon X \multimap X$ such that $\eta(x) \subseteq \varphi(x)$ for all $x \in X$.

Now we present an example of an s-admissible map which is neither admissible nor an s-map:

Example 4.11. Let $\varphi\colon [0,3] \multimap [0,3]$ be given by:

$$\varphi(x) = \begin{cases} [-x+1, -x+2] & \text{for } x \in [0,1); \\ [0, -x+2] \cup [-x+4, 3] & \text{for } x \in [1,2]; \\ [-x+4, -x+5] & \text{for } x \in (2,3]. \end{cases}$$

The map φ is not an s-map. Moreover, φ is not admissible, because the graph of φ has two connected components and neither of them is a graph of a multivalued map from $[0,3]$ to $[0,3]$. On the other hand we have an s-map $\eta\colon [0,3] \multimap [0,3]$ given by:

$$\eta(x) = \begin{cases} -x+1 & \text{for } x \in [0,1); \\ \{0,3\} & \text{for } x = 1; \\ -x+4 & \text{for } x \in (1,3] \end{cases}$$

such that $\eta(x) \subseteq \varphi(x)$ for all $x \in X$. Consequently φ is s-admissible. We have $\mathcal{L}_s(\eta) = 2$, so $2 \in \mathcal{L}_s(\varphi)$. Moreover, it can be shown that $\mathcal{L}_s(\varphi) = \{2\}$.

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Rational homotopy type of free and pointed mapping spaces between spheres

1 Introduction

Denote by $\mathrm{map}(X, Y)$ (respec. $\mathrm{map}^*(X, Y)$) the space of free (respec. pointed) maps from X to Y. Whenever X is a finite CW-complex and Y is a nilpotent CW-complex of finite type over $\mathbb{Q}$, then [8] any path component of both $\mathrm{map}(X, Y)$ and $\mathrm{map}^*(X, Y)$ are nilpotent CW-complexes of finite type over $\mathbb{Q}$ and in particular, it can be rationalized in the classical sense. From the Sullivan approach to rational homotopy theory [9], and based in the fundamental work of Haefliger [7], there is a standard procedure [2, 3] to obtain Sullivan models of the path components $\mathrm{map}_f(X, Y)$ and $\mathrm{map}^*_f(X, Y)$ of $\mathrm{map}(X, Y)$ and $\mathrm{map}^*(X, Y)$ respectively, containing the map $f\colon X \to Y$. In this note, we show the advantage of this procedure and use it repeatedly to explicitly describe the rational homotopy type of free and pointed mapping spaces between spheres:

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Theorem 1.1. *(i) For m odd and any $n \geq 1$,*

$$map(S^n, S^m) \simeq_{\mathbb{Q}} \begin{cases} S^m \times K(\mathbb{Z}, m-n), & \text{if } m > n. \\ \bigcup_{\mathbb{N}} S^m, & \text{if } m = n. \\ S^m, & \text{if } m < n. \end{cases}$$

$$map^*(S^n, S^m) \simeq_{\mathbb{Q}} \begin{cases} K(\mathbb{Z}, m-n), & \text{if } m > n. \\ \bigcup_{\mathbb{N}} *, & \text{if } m = n. \\ *, & \text{if } m < n. \end{cases}$$

(ii) For m even and any $n \geq 1$,

$$map(S^n, S^m) \simeq_{\mathbb{Q}} \begin{cases} Y, & \text{if } m > n. \\ S^m \times K(\mathbb{Z}, 2m-n-1) \bigcup_{\mathbb{N}} S^{2m-1}, & \text{if } m = n. \\ S^m \times K(\mathbb{Z}, 2m-n-1), & \text{if } m < n < 2m-1. \\ \bigcup_{\mathbb{N}} S^m, & \text{if } m = 2n-1. \\ S^m, & \text{if } m = 2n-1. \end{cases}$$

$$map^*(S^n, S^m) \simeq_{\mathbb{Q}} \begin{cases} K(\mathbb{Z}, m-n) \times K(\mathbb{Z}, 2m-n-1), & \text{if } m > n. \\ \bigcup_{\mathbb{N}} K(\mathbb{Z}, 2m-n-1), & \text{if } m = n. \\ K(\mathbb{Z}, 2m-n-1), & \text{if } m < n < 2m-1. \\ \bigcup_{\mathbb{N}} *, & \text{if } m = 2n-1. \\ *, & \text{if } m < n. \end{cases}$$

Here, $\simeq_{\mathbb{Q}}$ means "rationally homotopy equivalent"; $\bigcup$ denotes the disjoint union; and Y is a rational space which sits in a fibration of the form

$$S^m_{\mathbb{Q}} \times K(\mathbb{Q}, m-n) \to Y \to K(\mathbb{Q}, 2m-n-1).$$

We should mention that the above result might be known, or easily deduced by specialists. However, to our knowledge, it has not been made explicit in the literature. Thus, this paper reviews in a particular a useful situation, the general procedure of obtaining the rational homotopy type of both free and pointed mappping spaces.

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2 Models of mapping spaces between spheres

In this section we prove the theorem above. We highly depend on known facts and techniques arising from rational homotopy theory. All of them can be found in the excellent reference [6] which is now standard on the subject. Here, we simply present a summary of some basic facts.

For any simply connected, or more generally, nilpotent CW-complex of finite type X, its rationalization $X_{\mathbb{Q}}$ is a rational space (i.e., its homotopy groups are rational vector spaces), together with a map $X \to X_{\mathbb{Q}}$ inducing isomorphisms in rational homotopy.

On the other hand, to any space X there corresponds, in a contravariant way, its *minimal Sullivan model* which is a particular *Sullivan algebra* $(\Lambda V, d)$, unique up to isomorphism, which algebraically models the rational homotopy type of the space X, or equivalently, the homotopy type of its rationalization $X_{\mathbb{Q}}$. By ΛV we mean the free commutative algebra generated by the graded vector space V, i.e., $\Lambda V = TV/I$ where TV denotes the tensor algebra over V and I is the ideal generated by $v \otimes w - (-1)^{|w||v|} w \otimes v$, $\forall v, w \in V$. The differential d satisfies a certain minimality condition which, in the simply connectid case it translates to: for any element of $v \in V$, dv is a polynomial in ΛV with no linear term.

This correspondence yields an equivalence between the homotopy categories of 1-connected rational spaces of finite type and that of 1-connected rational commutative differential graded algebras of finite type. Indeed, this equivalence is the restriction to the appropriate subcategories of the classical adjoint functors [1]

$$\mathrm{SimplSets} \underset{\langle\,\rangle}{\overset{A_{PL}}{\rightleftarrows}} \mathrm{CDGA}$$

between the homotopy categories of commutative differential graded algebras and simplicial sets.

One can precise, through these functors, the notion of models of non connected spaces. As in [3], a *model* of a general space X, not necessarily connected, is a $\mathbb{Z}$-graded free CDGA $(\Lambda W, d)$ such that its simplicial realization $\langle(\Lambda W, d)\rangle$ has the same homotopy tye of the Milnor simplicial approximation of $X_{\mathbb{Q}}$, $S_*(X_{\mathbb{Q}})$.

We now introduce the Haefliger model [7] of the free and pointed mapping spaces $\mathrm{map}(X, Y)$, $\mathrm{map}^*(X, Y)$, via the functorial description of Brown-Szczarba [2].

Let B be a finite dimensional CDGA (commutative differential graded algebra) model of the finite CW-complex X and let $A = (\Lambda V, d)$ be a Sullivan model of the nilpotent CW-complex of finite type Y.

Denote by $B^\sharp = Hom(B, \mathbb{Q})$ the differential graded coalgebra, dual of B and therefore negatively graded, and consider the $\mathbb{Z}$-graded CDGA $\Lambda(A \otimes B^\sharp)$ with the natural differential induced by the one on A and by the dual δ of the differential on B. Now, consider the differential ideal $I \subset \Lambda(A \otimes B^\sharp)$ generated by $1 - 1 \otimes 1^\sharp$ and by the elements of the form

$$v_1 v_2 \otimes \beta - \sum_j (-1)^{|v_2||\beta'_j|} (v_1 \otimes \beta'_j)(v_2 \otimes \beta''_j),$$

with $v_1, v_2 \in V$, $\beta \in B$ and $\Delta\beta = \sum_j \beta'_j \otimes \beta''_j$. Then, the composition

$$\rho \colon \Lambda(V \otimes B^\sharp) \hookrightarrow \Lambda(A \otimes B^\sharp) \twoheadrightarrow \Lambda(A \otimes B^\sharp)/I$$

is an isomorphism of graded algebras [2, Thm.1.2]. Thus, we may consider on $\Lambda(V \otimes B^\sharp)$ the differential $\widetilde{d}$ for which the above becomes an isomorphisms of CDGA's. To explicitly determine $\widetilde{d}$ on the generator $v \otimes \beta \in V \otimes B^\sharp$, first compute $dv \otimes \beta + (-1)^{|v|} v \otimes \delta\beta$ and then use the relations which generate the ideal I to express $dv \otimes \beta$ as an element of $\Lambda(V \otimes B^\sharp)$.

Then, it turns out [2, Thm.1.3] that $\big(\Lambda(V \otimes B^\sharp), \widetilde{d}\big)$ is a model of $\mathrm{map}(X, Y)$ Moreover, if $B^\sharp_+$ denotes the subspace of $B^\sharp$ of strictly negative elements, $\big(\Lambda(V \otimes B^\sharp_+), \widetilde{d}\big)$ is a model of $\mathrm{map}^*(X, Y)$.

For the model of the components of $\mathrm{map}(X, Y)$ and/or $\mathrm{map}^*(X, Y)$ we follow the approach and notation of [3, 4]:

For any free CDGA $(\Lambda W, d)$, in which W is $\mathbb{Z}$-graded, and any algebra morphism $u \colon \Lambda W \longrightarrow \mathbb{Q}$ consider the differential ideal K_u generated by $A_1 \cup A_2 \cup A_3$, being

$$A_1 = W^{<0},\ A_2 = dW^0,\ A_3 = \{\alpha - u(\alpha) : \alpha \in W^0\}.$$

$(\Lambda W, d)/K_u$ is again a free CDGA of the form $(\Lambda(\overline{W}^1 \oplus W^{\geq 2}), d_u)$ in which $\overline{W}^1$ is a complement in W^1 of $d(W^0)$ modulo identifications via A_1 and A_3, see [3, §4] for details. Note that, $\overline{W}^1$ depends also on u. Moreover, if $(\Lambda W, d)$ is a model of a non-connected space X and u corresponds to a 0-simplex of X, as remarked in [2, 4.3], $(\Lambda(\overline{W}^1 \oplus W^{\geq 2}), d_u)$ is a Sullivan model of the path component of X containing the fixed 0-simplex.

Next, consider $\big(\Lambda(V\otimes B^\sharp), d\big)$ the model of $\operatorname{map}(X,Y)$ which we have just recalled and let $\varphi\colon (\Lambda V, d) \to B$ be a model of a given map $f\colon X \to Y$. The morphism φ clearly induces a natural augmentation which shall be denoted also by $\varphi\colon \big(\Lambda(V\otimes B^\sharp), \widetilde{d}\big) \to \mathbb{Q}$. Applying the process above to this particular case yields the Sullivan algebra

$$\big(\Lambda\big(\overline{V\otimes B^\sharp}^1 \otimes (V\otimes B^\sharp)^{\geq 2}\big), \widetilde{d}_\varphi\big)$$

which constitutes a Sullivan model of $\operatorname{map}_f(X,Y)$. In the same way,

$$\big(\Lambda\big(\overline{V\otimes B^\sharp_+}^1 \otimes (V\otimes B^\sharp_+)^{\geq 2}\big), \widetilde{d}_\varphi\big)$$

is a Sullivan model of $\operatorname{map}^*_f(X,Y)$.

To prove our Theorem we will apply all of the above to the particular case of choosing $X = S^m$ and $Y = S^n$ to be spheres, $m,n \geq 1$. For it, recall that, if m is an odd integer, the minimal model of S^m is the exterior algebra on a generator of degree m with zero differential $(\Lambda x_m, 0)$. On the other hand, if m is even, the minimal model of S^m is $(\Lambda x_m, y_{2m-1}, d)$, $dx_m = 0$, $dy_{2m-1} = x_m^2$. From now on, subscripts will always denote degree.

On the other hand, for any n, a coalgebra model of S^n is $B = \langle 1, \alpha_n\rangle$, in which α_n is a primitive cycle of degree $-n$, i.e., $\Delta\alpha_n = \alpha_n \otimes 1 + 1 \otimes \alpha_n$.

We will now distinguish different cases:

Case 1: m odd.

A model of $\operatorname{map}(S^n, S^m)$ is therefore,

$$(\Lambda(x_m \otimes 1, x_m \otimes \alpha_n), 0).$$

To avoid excessive notation we set $x_m \otimes 1 = a_m$ and $x_m \otimes \alpha_n = b_{m-n}$ and rewrite the above as:

$$(\Lambda(a_m, b_{m-n}), 0).$$

On the other hand, taking into account that the evaluation fibration

$$\operatorname{map}^*(S^n, S^m) \to \operatorname{map}(S^n, S^m) \to S^m$$

is modelled by

$$(\Lambda a_m, 0) \to (\Lambda(a_m, b_{m-n}), 0) \to (\Lambda b_{m-n}, 0),$$

a model for $\operatorname{map}^*(S^n, S^m)$ is simply $(\Lambda b_{m-n}, 0)$.

We now obtain Sullivan models for the path components and identify the homotopy type of their realizations.

Case 1.1: free maps.

$\mathbf{m > n}$:

In this case $(\Lambda(a_m, b_{m-n}), 0)$ is already a Sullivan model as b_{m-n} has positive degree. Hence the only component of $\mathrm{map}(S^n, S^n)$ has the rational homotopy type of the product $S^m \times K(\mathbb{Z}, m-n)$ of S^m with the Eilenberg-MacLane space of type $(\mathbb{Z}, m-n)$.

$\mathbf{m = n}$:

In this case b_{m-n} has degree 0 and there are a countable number of non homotopic morphisms $\varphi_\lambda \colon (\Lambda(a_m, b_{m-n}), 0) \to \mathbb{Q}$, one for each $\lambda \in \mathbb{Q}$, sending b_{m-n} to 1. Then, the procedure above give rise to a countable number of components, just like in the integral case, each of which with Sullivan model $(\Lambda a_m, 0)$ whose realization is just $S^m_{\mathbb{Q}}$.

Observe, as in [5, Ex. 3], that in this case, since $\mathrm{map}(S^m, S^n)$ has infinitely many components, its rational homology in degree zero is infinite dimensional. Thus, its rational cohomology, also in degree zero, has uncountable dimension. This sharply contrasts with the rational cohomology of its model $(\Lambda(x_m \otimes 1, x_m \otimes \alpha_n), 0)$, which in degree zero has countable dimension. This illustrates why, in the non-connected case, a model of a space does not preserve, in general, rational homotopy invariants.

$\mathbf{m < n}$:

In this case b_{m-n} has negative degree and therefore, it vanishes when considering models of components. Therefore, there is only one component with Sullivan model $(\Lambda a_m, 0)$ whose realization is again $S^m_{\mathbb{Q}}$.

Case 1.2: pointed maps.

$\mathbf{m > n}$:

As in this case b_{m-n} is of positive degree there is only one component with Sullivan model $(\Lambda b_{m-n}, 0)$ whose realization is $K(\mathbb{Q}, m-n)$.

$\mathbf{m = n}$:

As in the free case, there are a countable number of non homotopic morphisms $\varphi_\lambda \colon (\Lambda b_{m-n}, 0) \to \mathbb{Q}$, one for each $\lambda \in \mathbb{Q}$, sending b_{m-n} to λ. Thus, when replacing b_{m-n} by λ we obtain $\mathbb{Q}$ as a model for the corresponding component and therefore, each component is rationally trivial.

m < n:

In this case b_{m-n} has negative degree so there is only one component which is rationally trivial.

Case 2: m even.

In this case, a model of $\mathrm{map}(S^n, S^m)$ is again computed via the methods above:

$$(\Lambda(x_m \otimes 1, y_{2m-1} \otimes 1, x_m \otimes \alpha_n, y_{2m-1} \otimes \alpha_n), d)$$

To avoid excessive notation, as before, we set $x_m \otimes 1 = a_m$, $y_{2m-1} \otimes 1 = c_{2m-1}$, $x_m \otimes \alpha_n = b_{m-n}$, $y_{2m-1} \otimes \alpha_n = z_{2m-n-1}$ and rewrite this model as:

$$(\Lambda(a_m, c_{2m-1}, b_{m-n}, z_{2m-n-1}), d),$$

in which the differential is given by

$$da_m = db_{m-n} = 0, \quad dc_{2m-1} = a_m^2, \quad dz_{2m-n-1} = 2a_m b_{m-n}.$$

Concerning pointed maps and taking into account that the evaluation fibration

$$\mathrm{map}^*(S^n, S^m) \to \mathrm{map}(S^n, S^m) \to S^m$$

is modelled by

$$(\Lambda(a_m, c_{2m-1}), d) \to (\Lambda(a_m, c_{2m-1}, b_{m-n}, z_{2m-n-1}), d), \to (\Lambda(b_{m-n}, z_{2m-n-1}), 0)$$

a model for $\mathrm{map}^*(S^n, S^m)$ is simply $(\Lambda(b_{m-n}, z_{2m-n-1}), 0)$.

Then, on components:

Case 2.1: free maps.

m > n:

In this case both b_{m-n}, z_{2m-n-1} have positive degrees so the above is already a Sullivan model. Hence, there is only one component whose realization is a space Y which fits in a fibration of the form

$$S^m_{\mathbb{Q}} \times K(\mathbb{Q}, m-n) \to Y \to K(\mathbb{Q}, 2m-n-1).$$

m = n:

Now z_{2m-n-1} has positive degree but b_{m-n} has degree zero and there are a countable number of non homotopic morphisms

$$\varphi_\lambda\colon (\Lambda(a_m, c_{2m-1}, b_{m-n}, z_{2m-n-1}), d) \to \mathbb{Q},$$

one for each $\lambda \in \mathbb{Q}$, sending b_{m-n} to λ. This gives rise to a countable number of components. If $\lambda \neq 0$ then the corresponding component has Sullivan minimal model $(\Lambda c_{2m-1}, 0)$ whose realization is S^{2m-1}. On the other hand, if $\lambda = 0$, the corresponding component has Sullivan minimal model $(\Lambda(a_m, c_{2m-1}, z_{2m-n-1}), d)$, with $dz_{2m-n-1} = 0$ whose realization is of the rational homotopy type of $S^m \times K(\mathbb{Z}, 2m-n-1)$.

m < n < 2m − 1:

Now b_{m-n} has negative degree but z_{2m-n-1} has positive degree. Thus, there is only one component with model $(\Lambda(a_m, c_{2m-1}, z_{2m-n-1}), d)$, with $dz_{2m-n-1} = 0$ whose realization is again $S^m_{\mathbb{Q}} \times K(\mathbb{Q}, 2m-n-1)$.

n = 2m − 1:

Here, b_{m-n} has negative and z_{2m-n-1} has degree zero. Hence, we have a countable number of components arising from the CDGA morphisms $\varphi_\lambda\colon (\Lambda(a_m, c_{2m-1}, b_{m-n}, z_{2m-n-1}), d) \to \mathbb{Q}$, one for each $\lambda \in \mathbb{Q}$, sending z_{2m-n-1} to λ. Each of them produces via the procedure above the same Sullivan model $(\Lambda(a_m, c_{2m-1}), d)$ whose realization is $S^m_{\mathbb{Q}}$.

n > 2m − 1:

In this case both b_{m-n}, z_{2m-n-1} have negative degrees. Hence, there is only one component with model $(\Lambda(a_m, c_{2m-1}), d)$ whose realization is $S^m_{\mathbb{Q}}$.

Case 2.2: pointed maps.

m > n:

In this case, both b_{m-n}, z_{2m-n-1} have positive degrees and the model $(\Lambda(b_{m-n}, z_{2m-n-1}), 0)$ is already minimal. Thus, there is one component rationally equivalent to $K(\mathbb{Z}, m-n) \times K(\mathbb{Z}, 2m-n-1)$.

m = n:

Now, z_{2m-n-1} has positive degree but b_{m-n} has degree zero. Thus, as in precedent cases, it can be replaced by any rational number giving rise to a countable number of components each of which with model $(\Lambda z_{2m-n-1}, 0)$, whose realization is $K(\mathbb{Q}, 2m-n-1)$.

$\mathbf{m < n < 2m - 1}$**:**

Here, z_{2m-n-1} has positive degree but b_{m-n} is of negative degree. Hence, there is only one component with model $(\Lambda z_{2m-n-1}, 0)$ whose realization is again $K(\mathbb{Q}, 2m-n-1)$.

$\mathbf{n = 2m - 1}$**:**

In this case b_{m-n} has negative degree and z_{2m-n-1} is of degree zero. Hence, in the procedure of obtaining components, b_{m-n} vanishes while z_{2m-n-1} is replaced by any rational number giving rise to a countable number of rationally trivial components.

$\mathbf{n > 2m - 1}$**:**

Finally, both b_{m-n}, z_{2m-n-1} have negative degrees and there is only one component which is rationally trivial.

Summarizing all of the above finishes the proof of our theorem.

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Periodicity generated by adding machines

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We show that a homeomorphism of the plane $\mathbb{R}^2$ with an invariant Cantor set $\mathbf{C}$, on which the homeomorphism acts as an adding machine, possesses periodic points arbitrarily close to $\mathbf{C}$. The existence of periodic points near an invariant Cantor set is related to a shape theory question whether a solenoid invariant in a flow defined on $\mathbb{R}^3$ must be contained in a larger movable invariant compactum.

1 Introduction

Let $\phi : X \to X$ be a homeomorphism ($\mathbb{Z}$-action) or a flow ($\mathbb{R}$-action) defined on a metric space X. A set $A \subset X$ invariant under ϕ is *Lyapunov stable* if for every neighborhood U of A there is a neighborhood V of A such that for every $p \in V$, the forward orbit of p is contained in U. J. Buescu and I. Stewart proved in [8] (see also [9]) that if $h : \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism with an invariant Lyapunov stable Cantor set $\mathbf{C}$, and $h_{|\mathbf{C}}$ is an adding machine, then every neighborhood of $\mathbf{C}$ contains a periodic orbit of h. The theorem was also proved by H. Bell and K. Meyer in [2]. In addition, the authors construct in this paper a specific example of a Lyapunov stable adding machine in $\mathbb{R}^2$ invariant under a C^1 homeomorphism h of $\mathbb{R}^2$ and show that the theorem does not hold for a homeomorphism H on $\mathbb{R}^3$ and a Lyapunov stable adding machine invariant under H. We give a simple proof that without the assumption of Lyapunov stability a weaker version of the theorem holds: Every neighborhood of $\mathbf{C}$ contains a periodic point of h. The proof bears a similarity to the proof of the Cartwright-Littlewood Theorem given Morton Brown in [7]. The Cartwright-Littlewood Theorem asserts that if planar continuum Δ does not separate the plane $\mathbb{R}^2$ and is invariant under an orientation preserving homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$, then h has a fixed point $p \in \Delta$.

Much earlier E.S. Thomas considered in [18] one-dimensional solenoids invariant in a C^1 flow on a 3-manifold. A solenoid in this case is the inverse limit of circles with bonding maps being group homomorphisms. If almost all bonding maps are of degree one, then the solenoid is said to be trivial. Assuming that the flow on a non-trivial solenoid is minimal, the Poincaré first-return map on a local cross-section of the solenoid is an adding machine. The flow restricted to an invariant set is *minimal* on this set if every orbit is dense in the set. In case of a solenoid, this is equivalent to the fact that there are no fixed points in the solenoid, i.e., the flow is non-singular.

A compact invariant set is *isolated* if in some compact neighborhood it is the largest invariant set. The notion applies to both homeomorphisms and flows. Thomas uses isolating blocks, considered by C. Conley and R. W. Easton in [10] and previously by T. Ważewski in [20], in order to establish an Alexander-Spanier cohomology exact sequence involving the solenoid. He then shows that an invariant non-trivial solenoid in a nonsingular flow on a 3-manifold is not isolated. M. Kulczycki proved in [14] that under certain conditions, a planar adding machine is not isolated.

2 Adding machine

For a sequence of integers $(k_1, k_2, k_3, \ldots)$, each greater than one, denote by $\mathbf{C}(k_1, k_2, k_3, \ldots)$, or shortly by $\mathbf{C}$, the Cantor set $\Pi_{n=1}^{\infty}\mathbb{Z}/k_n\mathbb{Z}$.

Definition 1. *An adding machine is a homeomorphism* $\alpha : \mathbf{C} \to \mathbf{C}$ *such that if*

$$\alpha(i_1, i_2, i_3, \ldots) = (j_1, j_2, j_3, \ldots)$$

then

1. *if there is an $m \geq 1$ such that $i_n = k_n - 1$ for $n < m$ and $i_m < k_m - 1$, then $j_n = 0$ for $n < m$, $j_m = i_m + 1$, and $j_n = i_n$ for $n > m$,*

2. *otherwise $j_m = 0$ for all m, i.e., if $i_m = k_m - 1$ for $m \geq 1$, then $j_m = 0$ for $m \geq 1$.*

The map α is an *adding machine with base* $(k_1, k_2, k_3, \dots)$ acting on $\mathbf{C}$. The Cantor set itself is ofter referred to as an adding machine; precisely, an adding machine is the pair $(\mathbf{C}, \alpha)$.

Definition 2. *Let α be an adding machine with base $(k_1, k_2, k_3, \dots)$ acting on $\mathbf{C}$. For a finite sequence of integers $i_1, \dots, i_n$, $0 \leq i_j < k_j$ for $j \leq n$, define a cylinder of length n as the set*

$$C_{i_1,\dots,i_n} = \{(x_1, x_2, \dots) \,|\, x_1 = i_1, \dots, x_n = i_n\}.$$

Note that the cylinder $C_{i_1,\dots,i_n}$ is invariant under α^s , where s is a multiple of the product $k_1 \cdots k_n$.

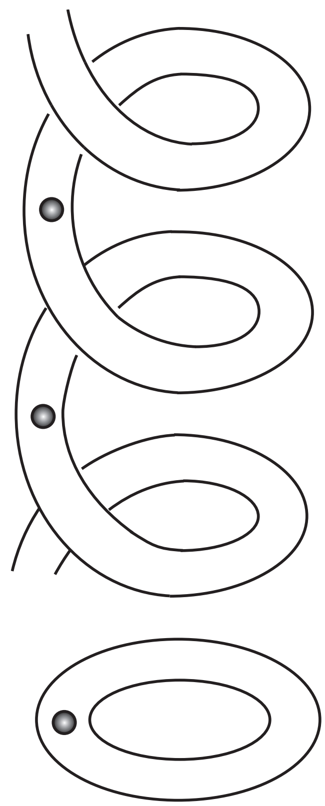
3 Periodic points near a planar adding machine

Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism and $\mathbf{C} = \mathbf{C}(k_1, k_2, k_3, \dots) \subset \mathbb{R}^2$ an invariant Cantor set. Assume that $h_{|\mathbf{C}}$ is an adding machine with base $(k_1, k_2, k_3, \dots)$. Let P be the set of periodic points of h in $\mathbb{R}^2$, including the fixed points although clearly the fixed points of h are away from C. Let $\mathrm{Cl}(P)$ be the closure of P. Each of the sets P and $\mathrm{Cl}(P)$ is invariant under h.

The theorem below shows that in every neighborhood of $\mathbf{C}$, there is a periodic point of h. Stability is not assumed.

Theorem. *$\mathbf{C} \cap \mathrm{Cl}(P) \neq \emptyset$.*

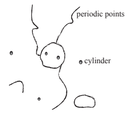
Proof. Suppose that $\mathbf{C} \cap \mathrm{Cl}(P) = \emptyset$. Let U be a component of $\mathbb{R}^2 \setminus \mathrm{Cl}(P)$ intersecting $\mathbf{C}$. Thus U contains a cylinder invariant under h^s, some power h^s of h. If U is simply connected, then by Brouwer's Theorem [6] we arrive at a contradiction that there is a fixed point of the orientation preserving homeomorphism $h^s \circ h^s$ in U, a periodic point of h outside P. The Brouwer Translation Theorem asserts that for a fixed point free,



orientation preserving homeomorphism of the plane no orbit of a point is bounded, hence there are no non-empty, compact, invariant sets.

In general, let $\widetilde{U}$ be the universal cover of U with $\pi : \widetilde{U} \to U$ the covering map. There is a cylinder $C_{i_1,\ldots,i_n}$ contained in an open, evenly covered disk $D \subset U$. Since $C_{i_1,\ldots,i_n}$ is invariant under $h^{k_1\cdots k_n}$, so is U. Let $f = h^{k_1\cdots k_n}_{|U}$. Since h has no periodic points in U, f as well as f^2, which is an orientation preserving homeomorphism of U, have no fixed points in U.

By composing a lift of f^2 with an appropriate deck transformation, we obtain an orientation preserving homeomorphism $\widetilde{f} : \widetilde{U} \to \widetilde{U}$ with an invariant compactum $\widetilde{C}$, a copy of $C_{i_1,\ldots,i_n}$ mapped homeomorphically by the projection π onto $C_{i_1,\ldots,i_n}$. Since $\widetilde{U}$ is homeomorphic to $\mathbb{R}^2$, by



Brouwer's theorem, $\widetilde{f}$ has a fixed point a. On the other hand since f^2 has no fixed points, no fiber $\pi^{-1}(p)$ is invariant under $\widetilde{f}$. Hence a cannot be a fixed point of $\widetilde{f}$.

Therefore the assumption that $\mathbf{C} \cap \mathrm{Cl}(P) = \emptyset$ is not valid. There are periodic points of h arbitrarily close to the Cantor set C. □

Remark. *The above theorem does not address the periods of the periodic points that are close to the Cantor set equipped with the adding machine with base* $(k_1, k_2, k_3, \ldots)$. *The almost periodicity of the adding machine yields natural relations of these periods to the products of numbers* $k_1, k_2, \ldots$ *multiplied by the number* 2 *in case of orientation reversing homeomorphisms.*

4 Shape theory

The notion of movability is one of the most important concepts of shape theory. A compact subset F of the Hilbert cube Q is *movable* [4] if for every neighborhood U of F there exists a neighborhood V of F such that for every neighborhood W of F there is a deformation of V into W within U. This property does not depend on the embedding of F in Q and the Hilbert cube can be replaced in the definition by any metric ANR. For the basic notions of the theory of shape the reader is referred to [3] and K. Borsuk's monograph [5]. The notion of movability seems closely related to notion of Lyapunov stability and thus it is of importance in dynamics.

Non-trivial solenoids were the first and most obvious examples of non-movable compacta. On the other hand, the Denjoy continua [11], which by construction are in a natural manner embedded in the surface of a torus, are movable. A description of a C^1 Denjoy set (conitnuum) is easily accessible in [15] or [16]. Denjoy continua are completely classified in [1] and [12]. Let D be a Denjoy continuum embedded in the surface of a torus $S^1 \times S^1$. Let $\pi : S^1 \times S^1 \to S^1 \times S^1$ be a covering projection with finite fibers. The set $\pi^{-1}(D)$ is *Denjoy-like.* The complement of a

Denjoy continuum in $S^1 \times S^1$ is connected, whereas the complement of a Denjoy-like continuum in $S^1 \times S^1$ may have several components.

Let $\phi : \mathbb{R} \times M \to M$ be a non-singular flow on a 3-manifold M. Let Σ be a solenoid in M approximated in terms of the Hausdorff distance by a sequence of pairwise disjoint simple closed curves $\{C_n\}_{n=1}^{\infty}$ disjoint from the solenoid. It is easy to show that the compactum $X = \Sigma \cup \bigcup_{n=1}^{\infty} C_n$ is movable.

Question 1. *If a solenoid Σ is invariant under ϕ, is Σ contained in a larger movable compact set invariant under ϕ?*

The next question is a slight variation of Question 1.

Question 2. *If a solenoid Σ is invariant under ϕ and U is a neighborhood of Σ, is Σ contained in a larger movable compact set invariant under ϕ and contained in U?*

Question 3. *Could the larger movable invariant set in Questions 2 always consist of Σ and a sequence of invariant approximating circles?*

In [17], P. Šindelářová constructed a flow on $\mathbb{R}^3$ with an invariant non-movable one-dimensional continuum Ω. The continuum is not a solenoid, but maps continuously onto a non-trivial solenoid and therefore by [19] or [13] it is not movable. In Šindelářová's flow, Ω is approximated by invariant Denjoy-like continua $\{D_n\}_{n=1}^{\infty}$ and the union $\Omega \cup \bigcup_{n=1}^{\infty} D_n$ is movable.

Question 4. *Is every compact invariant set in flow on a 3-manifold contained in a movable invariant set?*

Question 5. *If a compactum Y is invariant under a flow ϕ on a 3-manifold and U is a neighborhood of Y, is Σ contained in a movable compact set invariant under ϕ and contained in U?*

Question 6. *Would Questions 4 and 5 pose a different challenge if one assumed that the flow ϕ on the non-movable invariant set were minimal?*

Let D be a Cantor set in $\mathbb{R}^2$ invariant under an orientation preserving homeomorphism $g : \mathbb{R}^2 \to \mathbb{R}^2$. By Brouwer's theorem, g has a fixed point $p \in \mathbb{R}^2$. (It is easy to construct an example such that $g_{|D}$ is a Denjoy homeomorphism and g has no periodic points other than one fixed point.) This suggest the following:

Question 7. *Let Z be a compact invariant set in a flow on $\mathbb{R}^3$ such that there exists a sequence of invariant Denjoy-like continua $\{D_n\}_{n=1}^{\infty}$ so that the union $Z \cup \bigcup_{n=1}^{\infty} D_n$ is movable. Does there exist an invariant simple closed curve? Do there exist invariant simple closed curves arbitrarily close to Z?*

Finally let's recall the main problem:

Question 8. *Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism and let $\mathbf{C}$ be a Cantor set invariant under h. If $h_{|\mathbf{C}}$ is an adding machine, does there exist a periodic orbit in every neighborhood of $\mathbf{C}$?*

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Spheres over finite rings and their polynomial maps

The paper [8] grew out of our attempt to describe all polynomial self-maps of the real and complex circle as well.

Introduction. The definition of the n-sphere $\mathbb{S}^n$ with $n \geq 0$ over the reals can be extended to arbitrary commutative and unitary rings R which leads to the n-sphere

$$\mathbb{S}^n(R) = \{(r_0, \dots, r_n) \in R^{n+1};\ r_0^2 + \cdots + r_n^2 = 1\}$$

over R. If R is finite then it is worthwhile to compute its cardinality $\sharp(\mathbb{S}^n(R))$. More generally, if $V(F_q)$ is an affine variety defined over a finite field F_q, we can not only consider the number $\sharp(V(Fq))$, but also $\sharp(V(F_{q^m}))$ for $m \geq 1$. These can be nicely encoded by the Hasse-Weil zeta function of V: $\zeta(V;X) = \exp(\sum_{m=1}^{\infty} \frac{\sharp(V(F_{q^m}))}{m} X^m) \in \mathbb{Q}[[X]]$ which satisfies a number of fundamental properties, known as the Weil conjectures, which are known to be true mainly by the work [6] of Deligne.

Like for $\mathbb{S}^1$, the circle $\mathbb{S}^1(R)$ is equipped in an abelian group structure. Further, $\mathbb{S}^1(-)$ is a functor from commutative and unitary rings into abelian group. In particular, for the field $\mathbb{Q}$ of rational numbers, points of $\mathbb{S}^1(\mathbb{Q})$ are determined by Pythagorean triples and $\mathbb{S}^1(\mathbb{Q})$ is dense in the circle $\mathbb{S}^1$. If R is a finite ring then $\mathbb{S}^1(R)$ is a finite abelian group and it is a natural problem to determine its structure.

In [9], the author considers the group structure in $\mathbb{S}^1(R)$, with R being a commutative and unitary ring, determines this structure in the case when R is either a finite field or the ring $\mathbb{Z}_m$ of integers modulo m, and describes the group structure on conic sections.

In particular, by [9], the group $\mathbb{S}^1(R)$ is cyclic provided R is a field or the ring $\mathbb{Z}_{p^k}$ of integers modulo p^k for a prime odd number p. Further, in

[9, p. 54] the author has stated: *The case $p = 2$ is particularly interesting (or nasty, depending on your point of view [oder lästig, je nachdem, wie man es sieht]).*

The aim of Section 1 is to simplify proofs of some results from [9], present their generalizations and state in Theorem 2.5:
If p is a prime and $k \geq 1$ then

$$\mathbb{S}^1(\mathbb{Z}_{p^k}) \cong \begin{cases} \mathbb{Z}^+_{p^{k-1}(p-1)}, & if\ p \equiv 1 \,(\mathrm{mod}\, 4); \\ \mathbb{Z}^+_{p^{k-1}(p+1)}, & if\ p \equiv 3 \,(\mathrm{mod}\, 4); \\ \mathbb{Z}^+_2, & if\ \ k = 1; \\ \mathbb{Z}^+_2 \oplus \mathbb{Z}^+_{2^2} \oplus \mathbb{Z}^+_{2^{k-2}}, & if\ \ k \geq 2. \end{cases}$$

The paper [8] grew out of our attempt to describe all polynomial self-maps of the real and complex circle as well. Then, some results from [11, 14, 15] on spheres and their polynomial maps into spheres over any field has been transfered. In virtue of Wood [14] (see also [5, Chapter 13]) a necessary condition for the existence of a non-constant polynomial map $\mathbb{S}^m \to \mathbb{S}^n$ of spheres for $m \geq n$ is that $2^{k+1} > m \geq n \geq 2^k$ for some $k \geq 0$. It was shown in [15] that from the homotopy point of view nothing is lost by complexifying the problem of which homotopy classes of maps of spheres contain a polynomial representative. Furthermore in virtue of [7] any complex polynomial self-map of $\mathbb{S}^2(\mathbb{C})$ yields a regular self-map of the sphere $\mathbb{S}^2$ in a canonical way. Then Loday [11] using algebraic and topological K-theory proved some results on polynomials maps into $\mathbb{S}^n$. For instance, every polynomial map from the torus $\mathbb{T}^n$ to $\mathbb{S}^n$ is null-homotopic if $n > 1$. For n even those results were extended in [3, 4] to regular and then in [5] to polynomial maps $\mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_k} \to \mathbb{S}^n$ with $n = n_1 + \cdots + n_k$ odd. Certainly, polynomial maps $\mathbb{S}^{m_1}(R) \times \cdots \times \mathbb{S}^{m_k}(R) \to \mathbb{T}^n(R)$ are worth to be studied from the algebraic point of view for any field R. We made use of the abelian group structure on the sphere $\mathbb{S}^1(R)$ to show in [8, Corollary 2.11] that for any polynomial self-map $f : \mathbb{S}^1(R) \to \mathbb{S}^1(R)$ there are $\alpha \in \mathbb{S}^1(R)$ and an integer n such that $f(z) = \alpha z^n$ for any $z \in \mathbb{S}^1(R)$ provided the field R is infinite. All polynomial maps $\mathbb{S}^{m_1}(R) \times \cdots \times \mathbb{S}^{m_k}(R) \to \mathbb{T}^n(R)$ are listed in [8] for any infinite field R.

Section 2 takes up the systematic study of spheres $\mathbb{S}^n(R)$ over a finite field R and polynomial maps $\mathbb{S}^{m_1}(R) \times \cdots \times \mathbb{S}^{m_k}(R) \to \mathbb{S}^{n_1}(R) \times \cdots \times \mathbb{S}^{n_l}(R)$ with $m_1, \ldots, m_k, n_1, \ldots, n_l \geq 0$. Theorem 3.2 shows the cardinality $\sharp(\mathbb{S}^n(R))$ of the n-sphere $\mathbb{S}^n(R)$:

If the characteristic $\chi(R) \neq 2$ then for any number $n \geq 1$ it holds:

$$\sharp\mathbb{S}^n(R) = \begin{cases} (\sharp R)^n - (\sharp R)^{\frac{n}{2}}\eta((-1)^{\frac{n}{2}}), & \text{if } n \text{ is even;} \\ (\sharp R)^n - (\sharp R)^{\frac{n-1}{2}}\eta((-1)^{\frac{n+1}{2}}) & \text{if } n \text{ is odd,} \end{cases}$$

where

$$\eta(1) = 1 \text{ and } \eta(-1) = \begin{cases} 1, & \text{if the equation } X^2+1=0 \text{ has a solution in} R; \\ -1 & \text{otherwise} \end{cases}$$

and Corollary 3.4 asserts that any such any map $\mathbb{S}^{m_1}(R)\times\cdots\times\mathbb{S}^{m_k}(R) \to \mathbb{S}^{n_1}(R) \times \cdots \times \mathbb{S}^{n_l}(R)$ is a polynomial one.

1. Circles over a finite ring. Let R be a commutative and unitary ring. The set

$$\mathbb{S}^1(R) = \{(r_0, r_1) \in R \times R;\ r_0^2 + r_1^2 = 1\}$$

is called the 1-*sphere* or the *circle* over R.

Observe that on $\mathbb{S}^1(R)$ there is an abelian group structure defined by $(r_0, r_1) \circ (r_0', r_1') = (r_0r_0' - r_1r_1', r_0r_1' + r_1r_0')$ for any points $(r_0, r_1), (r_0', r_1') \in \mathbb{S}^1(R)$. Writing $SO(2, R)$ for the group of special orthogonal 2×2-matrices over R, we may easily show

Remark 2.1. (1) For any commutative and unitary ring R there is an isomorphism of groups

$$\mathbb{S}^1(R) \cong SO(2, R)$$

determined by the assignment $(r_0, r_1) \mapsto \begin{pmatrix} r_0 & r_1 \\ -r_1 & r_0 \end{pmatrix}$ for $(r_0, r_1) \in \mathbb{S}^1(R)$.

(2) If R_1, R_2 are commutative and unitary rings then there is an isomorphism of groups $\mathbb{S}^1(R_1 \times R_2) \cong \mathbb{S}^1(R_1) \times \mathbb{S}^1(R_2)$.

Next, consider the quotient ring $R[i] = R[X]/(X^2+1)$, where i denotes the class of X in $R[X]/(X^2 + 1)$ and write $U(R)$ for the multiplicative group of R. Let $\chi(R)$ denote the characteristic of R. Then, we may state:

Proposition 2.2. *For any unitary ring R there is a group monomorphism $\mathbb{S}^1(R) \to U(R[i])$. Further:*

(1) *if* $\chi(R) = 2$ *then* $\mathbb{S}^1(R) = \{(1+r+s, r);\ r, s \in R \,\text{with}\, s^2 = 0\}$ *and there is a splitting short exact sequence*

$$0 \to R^+ \to \mathbb{S}^1(R) \to \tilde{R} \to 1,$$

where R^+ *is the additive group of* R *and the group* $\tilde{R} = \{s \in R; s^2 = 0\}$ *with* $s_1 \circ s_2 = s_1 + s_2 + s_1 s_2$ *for* $s_1, s_2 \in \tilde{R}$*;*

(2) *if* $i \in R$ *with* $i^2 = -1$ *then there is an exact sequence of abelian groups*

$$0 \to R_0 \to \mathbb{S}^1(R) \to U(R),$$

where $R_0 = \{r \in R;\ 2r = 0\}$*;*

(i) *if* $2 \in U(R)$ *then there a group isomorphism*

$$\mathbb{S}^1(R) \xrightarrow{\cong} U(R);$$

(ii) *if* $\chi(R) = 2$ *then there is a splitting short exact sequence*

$$0 \to R \to \mathbb{S}^1(R) \to R_1 \to 1,$$

where $R_1 = \{r \in R;\ r^2 = 1\}$*;*

(3) *if* $i \notin R$ *then there is an exact sequence*

$$1 \to \mathbb{S}^1(R) \to U(R[i]) \xrightarrow{\rho} U(R)$$

of abelian groups, where $\rho(r_0 + r_1 i) = r_0^2 + r_1^2$ *for* $r_0 + r_1 i \in U(R[i])$. *Further, if* R *is a finite field then* $U(R[i]) \xrightarrow{\rho} U(R)$ *is onto.*

Proof. Certainly, the map $\varphi : \mathbb{S}^1(R) \to U(R[i])$ given by $\varphi(r_0, r_1) = r_0 + r_1 i$ for $(r_0, r_1) \in \mathbb{S}^1(R)$ is a group monomorphism.

(1) Let $\chi(R) = 2$. If $r, s \in R$ with $s^2 = 0$ then $(1+r+s, r) \in \mathbb{S}^1(R)$. Conversely, if $(r_0, r_1) \in \mathbb{S}^1(R)$ then $r_0 = 1 + r_1 + (1 + r_0 + r_1)$ and $(1 + r_0 + r_1)^2 = 0$. Hence, $\mathbb{S}^1(R) = \{(1+r+s, r);\ r, s \in R \,\text{with}\, s^2 = 0\}$. Further, one can easily see that the map $\phi : R^+ \to \mathbb{S}^1(R)$ given by $\phi(r) = (1+r, r)$ for $r \in R$ is a group monomorphism. Write $\tilde{R} = \{s \in R; s^2 = 0\}$ and $s_1 \circ s_2 = s_1 + s_2 + s_1 s_2$ for $s_1, s_2 \in \tilde{R}$. Then, $(\tilde{R}, \circ)$ is an abelian group and the map $\rho : \mathbb{S}^1(R) \to \tilde{R}$ given by $\rho(1+r+s, r) = s$ for $(1+r+s, r) \in \mathbb{S}^1(R)$ is an epimorphism. The sequence

$$0 \to R^+ \xrightarrow{\phi} \mathbb{S}^1(R) \xrightarrow{\rho} \tilde{R} \to 0$$

is exact and the map $\rho' : \tilde{R} \to \mathbb{S}^1(R)$ given by $\rho'(s) = (1+s, 0)$ for $s \in \tilde{R}$ determines its splitting.

(2) Write $R_0 = \{r \in R;\, 2r = 0\}$. Then, the maps

$$\alpha : R_0 \to \mathbb{S}^1(R) \text{ and } \varphi : \mathbb{S}^1(R) \to U(R)$$

given by $\alpha(r) = (1+r, r)$ for $r \in R_0$ and $\varphi(r_0, r_1) = r_0 + r_1 i$ for $(r_0, r_1) \in \mathbb{S}^1(R)$ are group homomorphisms with $\operatorname{Ker}\alpha = \{0\}$ and $\operatorname{Im}\alpha = \operatorname{Ker}\varphi$. Notice that $r \in U(R)$ with $r + r^{-1} = 2s$ for some $s \in R$ implies $(s, -(r-s)i) \in \mathbb{S}^1(R)$ and $\varphi(s, -(r-s)i) = r$. Consequently,

$$\operatorname{Im}\varphi = \{r \in U(R);\, r + r^{-1} \in 2R\}.$$

(i) If $2 \in U(R)$ then $R_0 = \{0\}$ and $r + r^{-1} \in \operatorname{Im}\varphi$ for $r \in U(R)$. Hence, the map

$$\psi : U(R) \to \mathbb{S}^1(R)$$

given by $\psi(r) = (2^{-1}(r^{-1}+r), 2^{-1}(r^{-1}-r)i)$ for $r \in U(R)$ is the inverse of the $\varphi : \mathbb{S}^1(R) \to U(R)$ above.

(ii) If $\chi(R) = 2$ then $R_0 = R$, $\operatorname{Im}\varphi = \{r \in R;\, r^2 = 1\} = R_1$ and the short exact sequence

$$0 \to R^+ \to \mathbb{S}^1(R) \to R_1 \to 1$$

splits as an exact sequence of elementary 2-groups.

(3) Consider the group homomorphism $\rho : U(R[i]) \to U(R)$ given by $\rho(r_0 + r_1 i) = r_0^2 + r_1^2$ for $r_0 + r_1 i \in U(R[i])$. Then, $\operatorname{Ker}\rho = \mathbb{S}^1(R)$ and consequently we get the required short exact sequence $1 \to \mathbb{S}^1(R) \to U(R[i]) \to U(R)$.

Let now R be a finite field and define the group endomorphism $\pi : U(R) \to U(R)$ given by $\pi(r) = r^2$ for $r \in U(R)$. If $\chi(R) = 2$ then π is an automorphism and so $U(R[i]) \xrightarrow{\rho} U(R)$ is onto.

Now, suppose that $\chi(R) \neq 2$ and write $\sharp X$ for the cardinality of a finite set X. Notice that the group endomorphism $U(R) \to U(R)$ given by $r \mapsto r^2$ for $r \in U(R)$ leads to $\ker\pi \cong \mathbb{Z}_2$ and $\sharp\{r^2;\, r \in U(R)\} = \frac{\sharp U(R)}{2}$. Given $r \in U(R)$, we follow [10, Remark 6.25] to consider the sets $A = \{r_0^2;\, r_0 \in U(R) \cup \{0\}\}$ and $B = \{r - r_1^2;\, r_1 \in U(R) \cup \{0\}\}$. Then, $\sharp A = \sharp B = \frac{\sharp U(R)}{2} + 1$ and consequently $A \cap B \neq \emptyset$ which implies that $\rho(r_0 + r_1 i) = r$.

$\square$

Writing $\mathbb{Z}_m^+$ for the cyclic group with order m, we deduce (see [9, Korollar 6]):

Corollary 2.3. *If R is a finite field then there is an isomorphism of groups:*
(1) $\mathbb{S}^1(R) \simeq (\mathbb{Z}_2^+)^k$ *provided* $\sharp R = 2^k$ *and* $\chi(R) = 2$;

(2) $\mathbb{S}^1(R) \simeq \begin{cases} \mathbb{Z}^+_{\sharp R-1}, & if\ \sharp R \equiv 1 \,(\mathrm{mod}\, 4); \\ \mathbb{Z}^+_{\sharp R+1}, & if\ \sharp R \equiv 3 \,(\mathrm{mod}\, 4). \end{cases}$ *provided* $\chi(R) \neq 2$.

Proof. (1) follows directly from Proposition 2.2(2)(ii).

(2) If $\sharp R \equiv 1 \,(\mathrm{mod}\, 4)$ then $i \in R$ and by Proposition 2.2(2), we get an isomorphism $\mathbb{S}^1(R) \cong U(R)$. Hence, the well-known isomorphism $U(R) \cong \mathbb{Z}^+_{\sharp R-1}$ yields $\mathbb{S}^1(R) \cong \mathbb{Z}^+_{\sharp R-1}$.

If $\sharp R \equiv 3\,(\bmod\ 4)$ then, by Fermat Theorem on Sums of Two Squares, $i \notin R$. Then, by Proposition 2.2(3), there is an exact sequence $1 \to \mathbb{S}^1(R) \to U(R[i]) \to U(R) \to 1$ of abelian groups. Because R and $R[i]$ are finite fields, there are isomorphisms $U(R) \cong \mathbb{Z}_{\sharp R-1}$ and $U(R[i]) \cong \mathbb{Z}_{(\sharp R)^2-1}$. Consequently, we deduce $\mathbb{S}^1(R) \cong \mathbb{Z}^+_{\sharp R+1}$ and the proof is complete.

□

Let now $R = \mathbb{Z}_m$, the ring of integers modulo m. The primary factorization $m = p_1^{k_1} \cdots p_t^{k_t}$ yields an isomorphism of rings $\mathbb{Z}_m \xrightarrow{\cong} \mathbb{Z}_{p_1^{k_1}} \times \cdots \times \mathbb{Z}_{p_t^{k_t}}$. Because $\mathbb{S}^1(-)$ is a product preserving functor from unitary rings to abelian groups, we get an isomorphism

$$\mathbb{S}^1(\mathbb{Z}_m) \xrightarrow{\cong} \mathbb{S}^1(\mathbb{Z}_{p_1^{k_1}}) \times \cdots \times \mathbb{S}^1(\mathbb{Z}_{p_t^{k_t}})$$

and $\sharp\mathbb{S}^1(\mathbb{Z}_m) = \sharp\mathbb{S}^1(\mathbb{Z}_{p_1^{k_1}}) \cdots \sharp\mathbb{S}^1(\mathbb{Z}_{p_t^{k_t}})$. Hence, the problem of determining the structure of $\mathbb{S}^1(\mathbb{Z}_m)$ and $\sharp\mathbb{S}^1(\mathbb{Z}_n)$ has been reduced to the case of prime powers p^k. By the claim in [9, p. 54], the group $\mathbb{S}^1(\mathbb{Z}_{p^k})$ is cyclic provided p is an odd prime. A proof of that is presented below.

Lemma 2.4. *If p is a prime and $k \geq 1$ then*

$$U(\mathbb{Z}_{p^k}[i]) \cong \begin{cases} \mathbb{Z}^+_{p^{k-1}(p-1)} \oplus \mathbb{Z}^+_{p^{k-1}(p-1)}, & if\ p \equiv 1 \,(\mathrm{mod}\, 4); \\ \mathbb{Z}^+_{p^{k-1}} \oplus \mathbb{Z}^+_{p^{k-1}(p^2-1)}, & if\ p \equiv 3 \,(\mathrm{mod}\, 4); \\ \mathbb{Z}^+_2, & if\ p = 2\, and\, k = 1; \\ \mathbb{Z}^+_{2^2} \oplus \mathbb{Z}^+_{2^{k-2}} \oplus \mathbb{Z}^+_{2^{k-1}}, & if\ p = 2\, and\, k \geq 2. \end{cases}$$

Proof. First, let p be an odd prime. Recall the well-known the isomorphism $U(\mathbb{Z}_{p^k}) \cong ((p)+1) \oplus U(\mathbb{Z}_p) \cong \mathbb{Z}^+_{p^{k-1}(p-1)}$ stated in [13, Theorem 6.7], where (p) is the nilpotent principal ideal of $\mathbb{Z}_{p^k}$ generated by p.

Let $p \equiv 1 \,(\mathrm{mod}\ 4)$ and $i \in U(\mathbb{Z}_{p^k})$ with order four. Because $i \in \mathbb{Z}_{p-1}$ and -1 is the only element in $\mathbb{Z}_{p-1}$ with order two, we deduce that $i^2 = -1$. Consequently, $\mathbb{Z}_{p^k}[i] \cong \mathbb{Z}_{p^k} \times \mathbb{Z}_{p^k}$ and $U(\mathbb{Z}_{p^k}[i]) \cong U(\mathbb{Z}_{p^k}) \times U(\mathbb{Z}_{p^k}) \cong \mathbb{Z}^+_{p^k(p-1)} \oplus \mathbb{Z}^+_{p^k(p-1)}$.

If $p \equiv 3 \,(\bmod\ 4)$ then, by Fermat's Theorem on Sums of Two Squares, $i \notin \mathbb{Z}_{p^k}$. Given $r_0 + r_1 i \in \mathbb{Z}_{p^k}[i]$, we see that $r_0 + r_1 i \in U(\mathbb{Z}_{p^k}[i])$ if and only if $r_0^2 + r_1^2 \in U(\mathbb{Z}_{p^k})$ or equivalently, if and only if $r_0 \in U(\mathbb{Z}_{p^k})$ or $r_1 \in U(\mathbb{Z}_{p^k})$. Hence, $\mathbb{Z}_{p^k}[i]$ is a p-primary ring with the nilpotent principal prime ideal (p) and $\sharp(p) = p^{2(k-1)}$. Then, the residue filed $\mathbb{Z}_{p^k}[i]/(p) \cong \mathbb{Z}_{p^2}$ and in view of [2, Proposition 1], we deduce that $U(\mathbb{Z}_{p^k}[i]) \cong ((p) + 1) \oplus U(\mathbb{Z}_{p^2})$. Following the proof of [13, Theorem 6.7], we get $(1+p)^{p^{l-2}}, (1+pi)^{p^{l-2}} \not\equiv 1 \,(\bmod\ p^l)$ and $(1+p)^{p^{l-1}}, (1+pi)^{p^{l-1}} \equiv 1 \,(\bmod\ p^l)$ for $l \geq 2$. Because $\langle 1+p \rangle \cap \langle 1+pi \rangle = \{1\}$, we deduce a group isomorphism $((p)+1) \cong \langle 1+p \rangle \oplus \langle 1+pi \rangle \cong \mathbb{Z}^+_{p^{k-1}} \oplus \mathbb{Z}^+_{p^{k-1}}$. Consequently, we get that $U(\mathbb{Z}_{p^k}[i]) \cong \mathbb{Z}^+_{p^{k-1}} \oplus \mathbb{Z}^+_{p^{k-1}(p^2-1)}$.

Let now $p = 2$. First, it is obvious that $U(\mathbb{Z}_2[i]) = \{1, i\} \cong \mathbb{Z}_2$. Hence, we can assume that $k \geq 2$. Recall form [13, Theorem 5.44] that $U(\mathbb{Z}_{2^k}) \cong \langle 5 \rangle \oplus \langle -1 \rangle \cong \mathbb{Z}^+_{2^{k-2}} \oplus \mathbb{Z}^+_2$ for $k \geq 2$. Because $r_0 + r_1 i \in U(\mathbb{Z}_{2^k}[i])$ if any only if r_0 is odd and r_1 is even or *vise versa*, we get $\sharp U(\mathbb{Z}_{2^k}[i]) = 2^{2k-1}$. Further, $(1+2i)^{2^{l-2}} \equiv 2^{l-1} + 1 + 2^{l-1} i \,(\mathrm{mod}\, 2^l)$ for $l > 2$. This implies that 2^{k-1} is the order of $1 + 2i$. Next, the intersection of any two of the subgroups $\langle i \rangle$, $\langle 5 \rangle$ and $\langle 1+2i \rangle$ is the trivial group and $\sharp U(\mathbb{Z}_{2^k}[i]) = 2^{2k-1}$. Thus, we deduce that $U(\mathbb{Z}_{2^k}[i]) \cong \langle i \rangle \oplus \langle 5 \rangle \oplus \langle 1+2i \rangle \cong \mathbb{Z}_{2^2} \oplus \mathbb{Z}_{2^{k-2}} \oplus \mathbb{Z}_{2^{k-1}}$ for $k \geq 2$ and the proof is complete.

$\square$

Now, we are in a position to show the main result of this Section:

Theorem 2.5. *If p is a prime and $k \geq 1$ then*

$$\mathbb{S}^1(\mathbb{Z}_{p^k}) \cong \begin{cases} \mathbb{Z}^+_{p^{k-1}(p-1)}, & if\ p \equiv 1 \,(\mathrm{mod}\, 4); \\ \mathbb{Z}^+_{p^{k-1}(p+1)}, & if\ p \equiv 3 \,(\mathrm{mod}\, 4); \\ \mathbb{Z}^+_2, & if\ \ k = 1; \\ \mathbb{Z}^+_2 \oplus \mathbb{Z}^+_{2^2} \oplus \mathbb{Z}^+_{2^{k-2}}, & if\ \ k \geq 2. \end{cases}$$

Proof. (1) If $p \equiv 1 \,(\mathrm{mod}\ 4)$ then $i \in \mathbb{Z}_{p^k}$. Because $2 \in U(\mathbb{Z}_{p^k})$, by Proposition 2.2(2), the map $\rho : \mathbb{S}^1(\mathbb{Z}_{p^k}) \to U(\mathbb{Z}_{p^k})$ given by $\rho(r_0, r_1) = r_0 + r_1 i$ for $(r_0, r_1) \in \mathbb{S}^1(\mathbb{Z}_{p^k})$ is an isomorphism of groups. Thus,

$$\mathbb{S}^1(\mathbb{Z}_{p^k}) \cong U(\mathbb{Z}_{p^k}) \cong \mathbb{Z}^+_{p^{k-1}(p-1)}.$$

(2) If $p \equiv 3 \,(\mathrm{mod}\ 4)$ then $i \notin \mathbb{Z}_{p^k}$. Further, $U(\mathbb{Z}_{p^k}) \cong \mathbb{Z}^+_{p^{k-1}(p-1)}$ and, in view of Lemma 2.4, it holds $U(\mathbb{Z}_{p^k}[i]) \cong \mathbb{Z}^+_{p^{k-1}} \oplus \mathbb{Z}^+_{p^{k-1}(p^2-1)}$. Next, consider the map $\rho : U(\mathbb{Z}_{p^k}[i]) \to U(\mathbb{Z}_{p^k})$ defined in Proposition 2.2(2). Then, the restriction $\rho|_{\mathbb{Z}^+_{p^{k-1}}}$ is an isomorphism and, in view of Proposition 2.2(3), the restriction $\rho|_{\mathbb{Z}^+_{p^2-1}}$ is onto. Consequently, $\rho : U(\mathbb{Z}_{p^k}[i]) \to U(\mathbb{Z}_{p^k})$ is onto and the short exact sequence $1 \to \mathbb{S}^1(\mathbb{Z}_{p^k}) \to U(\mathbb{Z}_{p^k}[i]) \xrightarrow{\rho} U(\mathbb{Z}_{p^k}) \to 1$ from Proposition 2.2(3) yields $\mathbb{S}^1(\mathbb{Z}_{p^k}) \cong \mathbb{Z}^+_{p^{k-1}(p+1)}$.

(3) For the group homomorphism $\rho : U(\mathbb{Z}_{2^k}[i]) \to U(\mathbb{Z}_{2^k})$ given by $\rho(r_0 + r_1 i) = r_0^2 + r_1^2$ for $r_0 + r_1 i \in U(\mathbb{Z}_{2^k}[i])$, by Proposition 2.2(3), we get the short exact sequence

$$1 \to \mathbb{S}^1(\mathbb{Z}_{2^k}) \to U(\mathbb{Z}_{2^k}[i]) \xrightarrow{\rho} U(\mathbb{Z}_{2^k})$$

of abelian groups with $k \geq 1$.

Because $U(\mathbb{Z}_2) = \{1\}$, Lemma 2.4 yields that $\mathbb{S}^1(\mathbb{Z}_2) \cong U(\mathbb{Z}_2[i]) \cong \mathbb{Z}^+_2$. If $k \geq 2$ then by the proof of Lemma 2.4, we have that $U(\mathbb{Z}_{2^k}[i]) \cong \langle i \rangle \oplus \langle 5 \rangle \oplus \langle 1 + 2i \rangle \cong \mathbb{Z}_{2^2} \oplus \mathbb{Z}_{2^{k-2}} \oplus \mathbb{Z}_{2^{k-1}}$. Because $\rho(i) = 1$, $\rho(5) = 5^2$, $\rho(1 + 2i) = 5$ and $U(\mathbb{Z}_{2^k}) \cong \langle 5 \rangle \oplus \langle -1 \rangle \cong \mathbb{Z}^+_{2^{k-2}} \oplus \mathbb{Z}^+_2$, we deduce that $\mathrm{Im}\, \rho = \langle 5 \rangle \cong \mathbb{Z}^+_{2^{k-2}}$. Consequently, the exact sequence

$$1 \to \mathbb{S}^1(\mathbb{Z}_{2^k}) \to U(\mathbb{Z}_{2^k}[i]) \xrightarrow{\rho} \mathbb{Z}_{2^{k-2}} \to 1$$

yields $\mathbb{S}^1(\mathbb{Z}_{2^k}) \cong \mathbb{Z}^+_2 \oplus \mathbb{Z}^+_{2^2} \oplus \mathbb{Z}^+_{2^{k-2}}$ for $k \geq 2$ and the proof is complete. $\square$

2. Spheres over finite fields and their polynomial maps. Let R be a commutative and unitary ring. Then, we notice:

Remark 3.1. For any commutative and unitary ring R there is a bijection $\mathbb{S}^3(R) \cong SU(R[i])$ determined by the assignment

$$(r_0, r_1, r_2, r_3) \mapsto \begin{pmatrix} r_0 + r_1 i & r_2 + r_3 i \\ -r_2 + r_3 i & r_0 - r_1 i \end{pmatrix}$$

for $(r_0, r_1, r_2, r_3) \in \mathbb{S}^3(R)$. Consequently, $\mathbb{S}^3(R)$ inherits the group structure from $SU(R[i])$. Notice that $\mathbb{S}^2(R) \cong \{A \in SU(R[i]); \operatorname{tr}(A) = 0\}$ provided $2R = 0$, where $tr : SU(R[i]) \to R[i]$ is the trace function.

Notice that there is an embedding $R_0^n \hookrightarrow \mathbb{S}^n(R)$ given by

$$(r_0, \ldots, r_{n-1}) \mapsto (1 + r_0 + \cdots + r_{n-1}, r_0, \ldots, r_{n-1})$$

for $(r_0, \ldots, r_{n-1}) \in R_0^n$, where $R_0 = \{r \in R; 2r = 0\}$. In particular, $R^n \hookrightarrow \mathbb{S}^n(R)$ provided $\chi(R) = 2$. If R is a field with $\chi(R) = 2$ then certainly there is a bijection $\mathbb{S}^n(R) \cong R^n$ and $\sharp\mathbb{S}^n(R) = (\sharp R)^n$.

Now, suppose that R is a finite field with $\chi(R) \neq 2$. Basing on [10, Theorems 6.26 and 6.27], we obtain:

Theorem 3.2. *If R is a finite field with $\chi(R) \neq 2$ then for any number $n \geq 1$ it holds:*

$$\sharp\mathbb{S}^n(R) = \begin{cases} (\sharp R)^n + (\sharp R)^{\frac{n}{2}} \eta((-1)^{\frac{n}{2}}), & \text{if } n \text{ is even;} \\ (\sharp R)^n - (\sharp R)^{\frac{n-1}{2}} \eta((-1)^{\frac{n+1}{2}}), & \text{if } n \text{ is odd,} \end{cases}$$

where $\eta(1) = 1$ *and* $\eta(-1) = \begin{cases} 1, & \text{if the equation } x^2 + 1 = 0 \\ & \text{has a solution in } R; \\ -1, & \text{otherwise.} \end{cases}$

Let $\sharp R = p^k$ for an odd prime p. Notice that $\eta(-1) = 1$ if and only if $p \equiv 1 \pmod 4$ or k is an even number.

To examine polynomial maps $P = (P_0, \ldots, P_n) : \mathbb{S}^m(R) \to \mathbb{S}^n(R)$ in that case a general result would be useful.

Proposition 3.3. *Let R be a field and $S \subseteq R^{m+1}$, $T \subseteq R^{n+1}$ finite subsets. Then any map $f : S \to T$ is a polynomial one for $m, n \geq 0$.*

Proof. Given a finite subset $S \subseteq R^{m+1}$ there is obviously a finite subset $S_0 = \{r_1, \ldots, r_k\} \subseteq R$ with $S \subseteq S_0^{m+1}$. It is well-know that there are interpolation polynomials $P_{r_1}(X), \ldots, P_{r_k}(X) \in R[X]$ with $P_{r_i}(x_j) =$

$\delta_{r_i r_j}$ for $i, j = 0, \ldots, k$. Next for any $s = (r_{i_0}, \ldots, r_{i_m}) \in S_0^{m+1}$ consider the polynomial

$$P_s(X_0, \ldots, X_m) = P_{r_{i_0}}(X_0) \cdots P_{r_{i_m}}(X_m) \in R[X_0, \ldots, X_m].$$

Then $P_s(s') = \delta_{ss'}$ for any $s, s' \in S_0^{m+1}$.

Now, given a map $f : S \to T$ write $f(s) = (f_0(s), \ldots, f_n(s))$ for any point $s \in S$. Then, the polynomial map $S \to T$ determined by polynomials:

$$Q_0(X_0, \ldots, X_m) = \sum_{s \in S} f_0(s) P_s(X_0, \ldots, X_m),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$Q_n(X_0, \ldots, X_m) = \sum_{s \in S} f_n(s) P_s(X_0, \ldots, X_m)$$

coincides with $f : S \to T$ and the proof is complete. $\square$

In particular, the following conclusion follows.

Corollary 3.4. *Let R be a finite field. Then any map $\mathbb{S}^{m_1}(R) \times \cdots \times \mathbb{S}^{m_k}(R) \to \mathbb{S}^{n_1}(R) \times \cdots \times \mathbb{S}^{n_l}(R)$ is a polynomial one for $m_1, \ldots, m_k, n_1, \ldots, n_l \geq 0$.*

Let $\mathrm{End}_R(R[X_1, \ldots, X_n])$ be the set of all R-homomorphisms of $R[X_1, \ldots, X_n]$ and $\mathrm{Aut}_R(R[X_1, \ldots, X_n])$ the group of all its R-automorphisms. Write $T(R, n)$ for the tame polynomial automorphism subgroup of $\mathrm{Aut}_R(R[X_1, \ldots, X_n])$ generated by $(X_1 + F(X_2, \ldots, X_n), X_2, \ldots, X_n)$ for all $F(X_2, \ldots, X_n) \in R[X_2, \ldots, X_n]$, $\mathcal{P}(R^n)$ for the set of all self-maps of R^n and $\mathcal{B}(R^n)$ the group of all bijections of R^n. Then, we get an obvious map

$$\mathcal{E} : \mathrm{End}_R(R[X_1, \ldots, X_n]) \longrightarrow \mathcal{P}(R^n).$$

Theorem 3.5. ([12]) *Let R be a finite field and F_p the simple field, where p is a prime. Then:*

(1) *$\sharp\mathcal{E}(T(R,1)) = \sharp\mathcal{B}(R)/\sharp R - 2)!$, so $\mathcal{E}(T(R,1)) = \mathcal{B}(R)$ only if $R = F_2, F_3$;*

(2) *if $n \geq 2$ and $\chi(R) \neq 2$ or $R = F_2$ then $\mathcal{E}(T(R,n)) = \mathcal{B}(R^n)$;*

(3) *if $n \geq 2$, $\chi(R) = 2$ and $\sharp R > 2$ then $\sharp\mathcal{E}(T(R,n) = \sharp\mathcal{B}(R^n)/2$. In fact,*

$\mathcal{E}(T(R,n))$ is the alternating subgroup $\mathcal{A}(R^n)$ of the group $\mathcal{B}(R^n)$.

Now, any bijection of $\mathbb{S}^{n_1}(R) \times \cdots \times \mathbb{S}^{n_l}(R)$ yields an bijection of $R^{m_1+\cdots+m_k+k}$. Furthermore, for $\chi(R) = 2$ there is an obvious polynomial isomorphism $\mathbb{S}^n(R) \to R^n$. Consequently, Theorem 3.5 leads to:

Corollary 3.6. *Let R be a finite field. Then:*

(1) *if $\chi(R) \neq 2$ or $R = F_2$ then any bijection of $\mathcal{B}(\mathbb{S}^{n_1}(R) \times \cdots \times \mathbb{S}^{n_l}(R))$ is an invertible polynomial map;*

(2) *if $\sharp R > 2$ and $\chi(R) = 2$ then any bijection of $\mathcal{A}(\mathbb{S}^{n_1}(R) \times \cdots \times \mathbb{S}^{n_l}(R))$ is an invertible polynomial map.*

Let R be a commutative and unitary ring. Then, we could consider the non-commutative and unitary ring $R\{i,j,k\}$ with $i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j$. Given $q = r_0 + r_1 i + r_2 j + r_3 k \in R\{i,j,k\}$, we write $|q|^2 = r_0^2 + r_1^2 + r_2^2 + r_3^2$ and $\bar{q} = r_0 - r_1 i - r_2 j - r_3 k$. Then, $q\bar{q} = |q|^2$, $|q_1 q_2|^2 = |q_1|^2 |q_2|^2$ for $q, q_1, q_2 \in R\{i,j,k\}$ and

$$\mathbb{S}^3(R) \cong \{q \in R\{i,j,k\};\ |q|^2 = 1\}.$$

Hence, $\mathbb{S}^3(R)$ inherits the group structure which coincides with the previous one. Further, we have a group monomorphism

$$\varphi : \mathbb{S}^3(R) \to U(R\{i,j,k\})$$

given by $\varphi(r_0, r_1, r_2, r_3) = r_0 + r_1 i + r_2 j + r_3 k$ for $(r_0, r_1, r_2, r_3) \in \mathbb{S}^3(R)$. Notice that $r_0 + r_1 i + r_2 j + r_3 k \in U(R\{i,j,k\})$ if and only if $r_0^2 + r_1^2 + r_2^2 + r_3^2 \in U(R)$. Hence, the map

$$\rho : U(R\{i,j,k\}) \to U(R)$$

given by $\rho(r_0 + r_1 i + r_2 j + r_3 k) = r_0^2 + r_1^2 + r_2^2 + r_3^2$ for $r_0 + r_1 i + r_2 j + r_3 k \in U(R\{i,j,k\})$ is a well-defined group homomorphism and the sequence

$$1 \to \mathbb{S}^3(R) \xrightarrow{\varphi} U(R\{i,j,k\}) \xrightarrow{\rho} U(R)$$

is exact.

Next, we consider the non-associative and unitary ring $R\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$, where products $e_s e_t$ are defined by the Cayley algebra rules for $s, t = 1, 2, 3, 4, 5, 6, 7$. Given $c = r_0 + r_1 e_1 + r_2 e_2 + r_3 e_3 + r_4 e_4 + r_5 e_5 + r_6 e_6 + r_7 e_7 \in R\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$, write $|c|^2 = r_0^2 + r_1^2 + r_2^2 + r_3^2 + r_4^2 + r_5^2 + r_6^2 + r_7^2$. Then, $|c_1 c_2|^2 = |c_1|^2 |c_2|^2$ for $c_1, c_2 \in R\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ and

$$\mathbb{S}^7(R) \cong \{c \in R\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\};\ |c|^2 = 1\}$$

inherits a non-associative group structure.

Notice that we have a non-associative group monomorphism

$$\varphi : \mathbb{S}^7(R) \to U(R\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\})$$

given by $\varphi(r_0, r_1, r_2, r_3, r_4, r_5, r_6, r_7) = r_0 + r_1e_1 + r_2e_2 + r_3e_3 + r_4e_4 + r_5e_5 + r_6e_6 + r_7e_7$ for $(r_0, r_1, r_2, r_3, r_4, r_5, r_6, r_7) \in \mathbb{S}^7(R)$. Notice that $r_0 + r_1e_1 + r_2e_2 + r_3e_3 + r_4e_4 + r_5e_5 + r_6e_6 + r_7e_7 \in U(R\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\})$ if and only if $r_0^2 + r_1^2 + r_2^2 + r_3^2 + r_4^2 + r_5^2 + r_6^2 + r_7^2 \in U(R)$. Hence, the map

$$\rho : U(R\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}) \to U(R)$$

given by $\rho(r_0 + r_1e_1 + r_2e_2 + r_3e_3 + r_4e_4 + r_5e_5 + r_6e_6 + r_7e_7) = r_0^2 + r_1^2 + r_2^2 + r_3^2 + r_4^2 + r_5^2 + r_6^2 + r_7^2$ for $r_0 + r_1e_1 + r_2e_2 + r_3e_3 + r_4e_4 + r_5e_5 + r_6e_6 + r_7e_7 \in U(R\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\})$ is a well-defined non-associative group homomorphism and the sequence

$$1 \to \mathbb{S}^7(R) \xrightarrow{\varphi} U(R\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}) \xrightarrow{\rho} U(R)$$

is exact.

If R_1, R_2 are commutative and unitary rings then there is a bijection $\mathbb{S}^n(R_1 \times R_2) \cong \mathbb{S}^n(R_1) \times \mathbb{S}^n(R_2)$ for $n \geq 0$. Because the primary factorization $m = p_1^{k_1} \cdots p_t^{k_t}$ yields an isomorphism of rings $\mathbb{Z}_m \xrightarrow{\cong} \mathbb{Z}_{p_1^{k_1}} \times \cdots \times \mathbb{Z}_{p_t^{k_t}}$, we derive a bijection

$$\mathbb{S}^n(\mathbb{Z}_m) \cong \mathbb{S}^n(\mathbb{Z}_{p_1^{k_1}}) \times \cdots \times \mathbb{S}^n(\mathbb{Z}_{p_t^{k_t}}).$$

Thus, the study of $\mathbb{S}^n(\mathbb{Z}_m)$ reduces to $\mathbb{S}^n(\mathbb{Z}_{p^k})$ for any prime p and $k \geq 1$.

Proposition 3.7. *If p is a prime and $k \geq 1$ then:*

$$(1)\ \sharp\mathbb{S}^3(\mathbb{Z}_{p^k}) = \begin{cases} p^{3k-2}(p^2-1), & \textit{if } p \textit{ is an odd prime;} \\ 2^{3k}, & \textit{if } p = 2; \end{cases}$$

$$(2)\ \sharp\mathbb{S}^7(\mathbb{Z}_{p^k}) = \begin{cases} p^{7k-4}(p^2-1)(p^2+1), & \textit{if } p \textit{ is an odd prime;} \\ 2^{7k}, & \textit{if } p = 2. \end{cases}$$

Proof. (1) First, notice that $r_0 + r_1i + r_2j + r_3k \notin U(\mathbb{Z}_{p^k}\{i, j, k\})$ if only if $r_0^2 + r_1^2 + r_2^2 + r_3^2 \equiv 0 \pmod p$ or equivalently, $r_0^2 + r_1^2 + r_2^2 + r_3^2 = 0$ in the field $\mathbb{Z}_p$.

If p is an odd prime then, in view of [10, Theorem 6.26], the equation $r_0^2 + r_1^2 + r_2^2 + r_3^2 = 0$ has $p^3 + (p-1)p$ solutions in $\mathbb{Z}_p$. Consequently, the equation $r_0^2+r_1^2+r_2^2+r_3^2\equiv 0\,(\mathrm{mod}\,p)$ has $p^{4(k-1)}(p^3+(p-1)p)=p^{4k-3}(p^2+p-1)$ solutions in $\mathbb{Z}_{p^k}$. This implies that $\sharp U(\mathbb{Z}_{p^k}\{i,j,k\})=p^{4k}-p^{4k-3}(p^2+p-1)=p^{4k-3}(p^2-1)(p-1)$.

If $p=2$ then the equation $r_0^2+r_1^2+r_2^2+r_3^2=0$ has 2^3 solutions in $\mathbb{Z}_2$. Consequently, the equation $r_0^2+r_1^2+r_2^2+r_3^2\equiv 0\,(\mathrm{mod}\,2)$ has $2^{4(k-1)}2^3=2^{4k-1}$ solutions in $\mathbb{Z}_{2^k}$. This implies that $\sharp U(\mathbb{Z}_{2^k}\{i,j,k\})=2^{4k}-2^{4k-1}=2^{4k-1}$.

Next, by Lagrange Four-Square Theorem, the map $\rho : U(\mathbb{Z}_{p^k}\{i,j,k\})\to U(\mathbb{Z}_{p^k})$ is onto for any prime p and $k\geq 1$. Hence, the short exact sequence

$$1\to \mathbb{S}^3(\mathbb{Z}_{p^k})\xrightarrow{\varphi} U(\mathbb{Z}_{p^k}\{i,j,k\})\xrightarrow{\rho} U(\mathbb{Z}_{p^k})\to 1$$

and $U(\mathbb{Z}_{p^k})\cong\begin{cases}\mathbb{Z}_{p^{k-1}(p-1)}, & \text{if } p \text{ is an odd prime;}\\ \{1\}, & \text{if } p=2 \text{ and } k=1;\\ \mathbb{Z}_2\oplus\mathbb{Z}_{2^{k-2}}, & \text{if } p=2 \text{ and } k\geq 2\end{cases}$
lead to (1).

(2) If p is an odd prime then, in view of [10, Theorem 6.26], the equation $r_0^2+r_1^2+r_2^2+r_3^2+r_4^2+r_5^2+r_6^2+r_7^2=0$ has $p^7+(p-1)p^3$ solutions in $\mathbb{Z}_p$. Consequently, the equation $r_0^2+r_1^2+r_2^2+r_3^2+r_4^2+r_5^2+r_6^2+r_7^2\equiv 0\,(\mathrm{mod}\,p)$ has $p^{8(k-1)}(p^7+(p-1)p^3)=p^{8k-5}(p^4+p-1)$ solutions in $\mathbb{Z}_{p^k}$. This implies that $\sharp U(\mathbb{Z}_{p^k}\{e_1,e_2,e_3,e_4,e_5,e_6,e_7\})=p^{8k}-p^{8k-5}(p^4+p-1)=p^{8k-5}(p^2-1)(p-1)(p^2+1)$.

If $p=2$ then the equation $r_0^2+r_1^2+r_2^2+r_3^2+r_4^2+r_5^2+r_6^2+r_7^2=0$ has 2^7 solutions in $\mathbb{Z}_2$. Consequently, the equation $r_0^2+r_1^2+r_2^2+r_3^2+r_4^2+r_5^2+r_6^2+r_7^2\equiv 0\,(\mathrm{mod}\,2)$ has $2^{8(k-1)}2^7=2^{8k-1}$ solutions in $\mathbb{Z}_{2^k}$. This implies that $\sharp U(\mathbb{Z}_{2^k}\{e_1,e_2,e_3,e_4,e_5,e_6,e_7\})=2^{8k}-2^{8k-1}=2^{8k-1}$.

Then, we follow *mutatis mutandis* the procedure presented in (1) and the proof is completed. $\square$

Now, for $z=r_0+r_1i\in R[i]$, we write $|z|^2=r_0^2+r_1^2$ and $\bar{z}=r_0-r_1i$. Then, $z\bar{z}=|z|^2$, $z\in U(R[i])$ if and only if $|z|^2\in U(R)$ and

$$\mathbb{S}^3(R)\cong\{(z_0,z_1)\in R[i]\times R[i];\ |z_0|^2+|z_1|^2=1\}.$$

Notice that there is an action

$$\circ : \mathbb{S}^1(R)\times\mathbb{S}^3(R)\longrightarrow\mathbb{S}^3(R)$$

such that $\lambda\circ(z_0,z_1)=(\lambda z_0,\lambda z_1)$ for $\lambda\in\mathbb{S}^1(R)$ and $(z_0,z_1)\in\mathbb{S}^3(R)$.

Next, $q \in U(R\{i,j,k\})$ if and only if $|q|^2 \in U(R)$ for $q \in R\{i,j,k\}$, and

$$\mathbb{S}^7(R) \cong \{(q_0, q_1) \in R\{i,j,k\} \times R\{i,j,k\};\ |q_0|^2 + |q_1|^2 = 1\}.$$

Further, there is an action

$$\circ : \mathbb{S}^3(R) \times \mathbb{S}^7(R) \longrightarrow \mathbb{S}^7(R)$$

such that $\lambda \circ (q_0, q_1) = (\lambda q_0, \lambda q_1)$ for $\lambda \in \mathbb{S}^3(R)$ and $(q_0, q_1) \in \mathbb{S}^7(R)$.

Now, we mimic the Hopf maps $h : \mathbb{S}^3 \longrightarrow \mathbb{S}^2$ and $H : \mathbb{S}^7 \longrightarrow \mathbb{S}^4$ to define

$$h(R) : \mathbb{S}^3(R) \longrightarrow \mathbb{S}^2(R)$$

by $h(R)(z_0, z_1) = (|z_0|^2 - |z_1|^2, 2z_0\bar{z}_1)$ for $(z_0, z_1) \in \mathbb{S}^3(R)$ and

$$H(R) : \mathbb{S}^7(R) \longrightarrow \mathbb{S}^4(R)$$

by $H(R)(q_0, q_1) = (|q_0|^2 - |q_1|^2, 2q_0\bar{q}_1)$ for $(q_0, q_1) \in \mathbb{S}^7(R)$.

Proposition 3.8. *Let R be a local commutative and unitary ring such that 2 is not a zero divisor of R. Then:*

$$(1)\ \ h(R)^{-1}(h(R)(z_0, z_1)) = \{(\lambda z_0, \lambda z_1);\ for\, \lambda \in \mathbb{S}^1(R)\} \cong \mathbb{S}^1(R)$$

for any $(z_0, z_1) \in \mathbb{S}^3(R)$;

$$(2)\ \ H(R)^{-1}(h(R)(q_0, q_1)) = \{(\lambda q_0, \lambda q_1);\ for\, \lambda \in \mathbb{S}^3(R)\} \cong \mathbb{S}^3(R)$$

for any $(q_0, q_1) \in \mathbb{S}^7(R)$.

Proof. (1) Let $(z_0, z_1) \in \mathbb{S}^3(R)$. Then, certainly it holds $\{(\lambda z_0, \lambda z_1); \text{for}\, \lambda \in \mathbb{S}^1(R)\} \subseteq h(R)^{-1}(h(R)(z_0, z_1))$.

Suppose that $h(R)(w_0, w_1) = h(R)(z_0, z_1)$ for some $(w_0, w_1) \in \mathbb{S}^3$. Then, $|w_0|^2 - |w_1|^2 = |z_0|^2 - |z_1|^2$ and $2w_0\bar{w}_1 = 2z_0\bar{z}_1$. Because $|w_0|^2 + |w_1|^2 = 1 = |z_0|^2 + |z_1|^2$ and $2 \in R$ is not a zero divisor, we get $|w_0|^2 = |z_0|^2$, $|w_1|^2 = |z_1|^2$ and $w_0\bar{w}_1 = z_0\bar{z}_1$. Further, R is a local ring, so $|w_0|^2 + |w_1|^2 = 1 = |z_0|^2 + |z_1|^2$ implies $|w_0|^2 \in U(R)$ or $|w_1|^2 \in U(R)$ and $|z_0|^2 \in U(R)$ or $|z_1|^2 \in U(R)$. Hence, $w_0 \in U(R)$ or $w_1 \in U(R)$ and $z_0 \in U(R)$ or $z_1 \in U(R)$.

If $z_0 \in U(R)$ then we set $\lambda = z_0^{-1}w_0$; if $z_1 \in U(R)$ then we set $\lambda = z_1^{-1}w_1$. Thus, $\lambda \in \mathbb{S}^1(R)$ and $(w_0, w_1) = (\lambda z_0, \lambda z_1)$. Because $(z_0, z_1) \in$

$\mathbb{S}^3(R)$ implies $z_0 \in U(R)$ or $z_1 \in U(R)$, we get $h(R)^{-1}(h(R)(z_0, z_1)) \cong \mathbb{S}^1(R)$.

(2) Given $(q_0, q_1) \in \mathbb{S}^7(R)$, we follow *mutatis mutandis* (1) to complete the proof.

□

By [1, Theorem 8.7], any commutative Artinian and unitary ring (in particular, any finite commutative and unitary ring) is a finite product of commutative Artinian local rings. Further, $\mathbb{S}^n(R_1 \times R_2) \cong \mathbb{S}^n(R_1)\times\mathbb{S}^n(R_2)$ for any commutative and unitary rings R_1, R_2 and $n \geq 0$. Consequently, in view of Proposition 3.8, for a commutative Artinian and unitary ring R, and such that 2 is not a zero divisor in R, we get embeddings

$$\bar{h}(R) : \mathbb{S}^3(R)/\mathbb{S}^1(R) \longrightarrow \mathbb{S}^2(R) \text{ and } \bar{H}(R) : \mathbb{S}^7(R)/\mathbb{S}^3(R) \longrightarrow \mathbb{S}^4(R).$$

In particular:

if R is a finite field with $\chi(R) \neq 2$ then Corollary 2.3 and Theorem 3.2 imply that $\bar{h}(R) : \mathbb{S}^3(R)/\mathbb{S}^1(R) \longrightarrow \mathbb{S}^2(R)$ and $\bar{H}(R) : \mathbb{S}^7(R)/\mathbb{S}^3(R) \longrightarrow \mathbb{S}^4(R)$ are bijections;

if $R = \mathbb{Z}_{p^k}$ for an odd prime p and $k \geq 1$ then Theorem 2.5 and Proposition 3.7 lead to:

$$\sharp\mathbb{S}^2(\mathbb{Z}_{p^k}) \geq \begin{cases} p^{3k-2}(p+1), & \text{if } p \equiv 1 \,(\mathrm{mod}\, 4); \\ p^{3k-2}(p-1), & \text{if } p \equiv 3 \,(\mathrm{mod}\, 4) \end{cases}$$

and

$$\sharp\mathbb{S}^4(\mathbb{Z}_{p^k}) \geq p^{4k-2}(p^2+1).$$

Remark 3.9. Because

$\mathbb{S}^{15}(R) \cong \{(c_0, c_1) \in R\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}\times R\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\};$

$|c_0|^2 + |c_1|^2 = 1\},$

we make use the Hopf map $\mathcal{H} : \mathbb{S}^{15} \to \mathbb{S}^8$ to consider $\mathcal{H}(R) : \mathbb{S}^{15}(R) \to \mathbb{S}^8(R)$ for a commutative and unitary ring R, and state a result as in Proposition 3.8 as well.

We close the paper with:

Conjecture 3.10. If p is an odd prime and $k \geq 1$ then:

$$(1)\ \sharp\mathbb{S}^2(\mathbb{Z}_{p^k}) = \begin{cases} p^{3k-2}(p+1), & \text{if } p \equiv 1 \,(\mathrm{mod}\, 4); \\ p^{3k-2}(p-1), & \text{if } p \equiv 3 \,(\mathrm{mod}\, 4); \end{cases}$$

$$(2)\ \sharp\mathbb{S}^4(\mathbb{Z}_{p^k}) = p^{4k-2}(p^2+1).$$

and

Problem 3.11. Let p be an odd prime and $k \geq 1$. Find:
(1) $\sharp(\mathbb{S}^n(\mathbb{Z}_{p^k}))$ for $n > 4$ with $n \neq 7$;
(2) the group structure of $\mathbb{S}^3(\mathbb{Z}_{p^k})$.

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The Borsuk-Ulam Theorem for Double Coverings of Seifert Manifolds

We study a Borsuk-Ulam type theorem for pairs (M, τ) *with* τ *a fixed point free involution of* M*, and such that both* M *and* $N\colon = M/\tau$ *are Seifert manifolds. In this note our point of view will be to start with a Seifert manifold* N*. Any non-trivial element* $\xi \in H^1(N; Z_2)$ *then gives rise to a pair* $(M_\xi, \tau_\xi) = (M, \tau)$ *with* M *(necessarily) also a Seifert manifold, and a double covering* $p\colon M \twoheadrightarrow N$*, with* τ *being the fixed point free involution on* M *associated to this double covering as the non-trivial deck transformation. We then seek the largest value of* n*, called the* $\mathbb{Z}_2$*-index of* (M, τ)*, such that the Borsuk-Ulam property holds for maps into* $\mathbb{R}^n$*, i.e. such that for every continuous map* $f\colon M \to \mathbb{R}^n$*, there is an* $x \in M$ *such that* $f(x) = f(\tau(x))$*. In case* M *is a* 3*-manifold (such as a Seifert manifold), the* $\mathbb{Z}_2$*-index can take only the values* 1, 2, 3.

1 Introduction

The study of involutions on manifolds has been of great interest and importance within topology, as illustrated by the books of J. Matoušek [12] and S. L. de Medrano [11] (and in particular, for involutions on Seifert manifolds, cf. the book of Montesinos [13]). The most famous theorem in the subject is undoubtedly the classical Borsuk-Ulam theorem, which applies to the antipodal involution of a sphere. This theorem together with various generalizations and applications continues to be of great interest. For example, a generalization of the Borsuk-Ulam theorem that applies to a fixed point free involution on any manifold has recently been studied by Gonçalves, Hayat, and Zvengrowski [7]. The case of manifolds of dimension 2 and the corresponding Borsuk-Ulam theorem has also been recently studied by Gonçalves and Guaschi [6]. The above mentioned book of Matoušek gives an extensive set of references related to the Borsuk-Ulam theorem; in addition to these further interesting aspects and generalizations of the classical Borsuk- Ulam theorem appear (among others) in work by K. D. Joshi [10], J. Jaworowski [9], A. Dold [5], and more recently in work of P. L. Q. Pergher, D. de Mattos, E. L. dos Santos [16], P. L. Q. Pergher, H. K. Singh, T. B. Singh [17], as well as survey papers among which we mention H. Steinlein [22], and I. Nagasaki [14].

In this paper we attempt to initiate this study for the Seifert manifolds, a large and important class of 3-manifolds introduced by Seifert [19] in 1933. This is possible, using the aforementioned paper [7] and the knowledge of the $\mathbb{Z}_2$-cohomology rings of these manifolds, cf. [2], [3], [4] for the orientable case and more recently [1] for all Seifert manifolds. We will suppose throughout that all manifolds under consideration are closed and connected.

Given a (closed, connected) m-manifold N, any non-trivial element $\xi \in H^1(N;\mathbb{Z}_2)$ gives rise to an epimorphism $\phi\colon \pi_1 N \twoheadrightarrow \mathbb{Z}_2$ and a pair $(M_\xi, \tau_\xi) = (M, \tau)$, where $p\colon M \twoheadrightarrow N$ is a double covering, M is a (closed, connected) m-manifold, and τ is the fixed point free involution on M associated to this double covering as the non-trivial deck transformation. This correspondence is via the sequence of isomorphisms

$$hom(\pi_1(N), \mathbb{Z}_2) \approx hom((\pi_1(N))_{ab}, \mathbb{Z}_2) \approx hom(H_1(N), \mathbb{Z}_2) \approx H^1(N;\mathbb{Z}_2). \tag{1}$$

Definition 1.1. *(i) We say that the Borsuk-Ulam property BU(n) holds for (M,τ) if for every continuous map $f\colon M \to \mathbb{R}^n$, there is an $x \in M$ such that $f(x) = f(\tau(x))$.*

(ii) The $\mathbb{Z}_2$-index $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau)$ is then defined as the largest $n \leq \infty$ such that BU(n) holds.

>From [7] it is known that $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau) \geq 1$ always holds, and $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau) = 1$ if and only if $\xi \in \mathrm{Im}(\rho\colon H^1(N;\mathbb{Z}) \to H^1(N;\mathbb{Z}_2)$, where ρ is the coefficient homomorphism induced by the surjection $\mathbb{Z} \twoheadrightarrow \mathbb{Z}_2$. Furthermore, it is shown there that $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau) \leq m = \dim(M)$ and $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau) = m$ if and only if $\xi^m \neq 0 \in H^m(N;\mathbb{Z}_2)$. It follows that the inequality $1 \leq \mathrm{ind}_{\mathbb{Z}_2}(M,\tau) \leq m$ is always satisfied. In particular, for $m = 3$, the $\mathbb{Z}_2$-index can only equal $1, 2$, or 3. These facts are formally stated in Section 2 as Theorem 2.1.

In the present work, we suppose that N is a Seifert manifold (of dimension $m = 3$), presented in the usual way by its Seifert invariants (cf. [15], [19]). The presentation of $\pi_1(N)$, associated to these invariants, is the standard presentation found in [15], and allows one to list the (non-trivial) homomorphisms $\phi\colon \pi_1 N \twoheadrightarrow \mathbb{Z}_2$. We classify the ϕ's for which the $\mathbb{Z}_2$-index equals 1, equals 2, or equals 3. The main results are expressed in terms of the Seifert invariants of N and the homomorphism ϕ.

This work contains five sections. In Section 2, we recall some basic facts about Seifert manifolds. In Section 3 we consider the situation of maps into $\mathbb{R}^2$; the main results are Proposition 3.4 and Theorem 3.5. The former gives necessary and sufficient conditions for $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau)=1$, and the latter (which is essentially the negation of the former) for $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau)\geq 2$. In Section 4 we consider the situation of maps into $\mathbb{R}^3$; the main result is Theorem 4.3 which gives necessary and sufficient conditions for $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau)=3$. In Section 5 we make some general comments about the relation between the $\mathbb{Z}_2$-index $=2$ and the $\mathbb{Z}_2$-index $=$ 3 cases. In this section we also study several specific examples that effectively illustrate the techniques, for a variety of Seifert manifolds, and also show that the distinction between the various cases can be surprisingly delicate.

Another (and probably more natural) approach to these questions is to start with the manifold M and fixed point free involution τ, then construct N as the orbit space M/τ. For Seifert manifolds M this can lead to cases that are not covered in the present paper, indeed cases where N is not a Seifert manifold in the classical sense, depending on the geometry (in the sense of Thurston) of M. The authors hope to complete the study, from this point of view, in subsequent research, with [7] being the first step in this direction and the present note the second step. We also note that the condition $\xi^m \neq 0$ mentioned above becomes $\xi^3 \neq 0$ for a 3-manifold, and for orientable 3-manifolds this condition also arises in the study of general relativity (where one says such 3-manfolds have "type 1"), cf. [21]. The condition $\xi^3 \neq 0$ is equivalent to the existence of a degree 1 (or odd degree) map of the 3-manifold onto $\mathbb{R}P^3$.

2 Introductory Remarks and Notation for 3-manifolds

Let N be a 3-manifold. In Section 1 the isomorphism (1) between $H^1(N;\mathbb{Z}_2)$ and $hom(\pi_1(N),\mathbb{Z}_2)$ was introduced. Under this correspondence, the image in $H^1(N;\mathbb{Z}_2)$ of a homomorphism $\phi\colon \pi_1(N)\to\mathbb{Z}_2$ will be denoted by $\xi_\phi = \xi$. Any non-zero element $\xi \in H^1(N;\mathbb{Z}_2)$ corresponds to an epimorphism $\phi\colon \pi_1 N \twoheadrightarrow \mathbb{Z}_2$ which induces a short exact sequence:

$$1 \to \operatorname{Ker}\phi \rightarrowtail \pi_1\mathrm{N} \twoheadrightarrow \mathbb{Z}_2 \to 0.$$

>From the theory of covering spaces, we know that there exists a connected 3-manifold $M = M_\phi$ such that $\operatorname{Ker}\phi = \pi_1(\mathrm{M})$ is a normal, index 2, subgroup of $\pi_1(N)$, and $M \twoheadrightarrow N$ is the regular double covering of N corresponding to $\operatorname{Ker}\phi$. We also know that the non-trivial deck transformation is a fixed point free involution $\tau_\phi = \tau$ on M such that the quotient M/τ is homeomorphic to N. We will use this correspondence freely whenever necessary.

From [7] Theorems (3.1) and (3.2) we have:

Theorem 2.1. *Let N be a 3-manifold and $\phi\colon \pi_1(N) \twoheadrightarrow \mathbb{Z}_2$ an epimorphism. Let (M,τ) and $\xi \in H^1(N;\mathbb{Z}_2)$ be determined as above.*

(i) One has $\operatorname{ind}_{\mathbb{Z}_2}(M,\tau) = 1$ if and only if the homomorphism $\phi\colon \pi_1(N) \twoheadrightarrow \mathbb{Z}_2$ factors through the projection $\mathbb{Z} \twoheadrightarrow \mathbb{Z}_2$ (equivalently $\xi \in \operatorname{Im}(\rho\colon H^1(N;\mathbb{Z}) \to H^1(N;\mathbb{Z}_2))$), otherwise $\operatorname{ind}_{\mathbb{Z}_2}(M,\tau) \in \{2,3\}$,

(ii) One has $\operatorname{ind}_{\mathbb{Z}_2}(M,\tau) = 3$ if and only if $\xi^3 \neq 0$.

We now focus on the situation where N is *any* Seifert manifold (orientable or not), as introduced in [19]. We shall answer the following question: given a presentation of N in terms of Seifert invariants, for which ϕ is $\operatorname{ind}_{\mathbb{Z}_2}(M,\tau) = 1, 2, \text{or } 3$?

Following the notation of Orlik [15], from now on, N will be a Seifert manifold described by a list of Seifert invariants

$$\{e; (\in, g); (a_1, b_1), \ldots, (a_n, b_n)\}$$

(note that Orlik uses b for the Euler number e). We do not need them to be "normalized" as in [15] and [19]: we only assume that e is an integer, the type $\in$ will described below, g is the genus of the base surface (the orbit space obtained by identifying each S^1 fibre of N to a point), and for each k, the integers a_k, b_k are coprime with $a_k \neq 0$ (in case $b_k = 0$ then $a_k = \pm 1$).

As in [15], p.74 (and elsewhere), it is convenient to add an additional (non-exceptional) fibre $a_0 = 1, b_0 = e$. We shall then use the following presentation of the fundamental group of N:

$$\pi_1(N) = \left\langle \begin{matrix} s_0, \ldots, s_n \\ v_1, \ldots, v_{g'} \\ h \end{matrix} \left| \begin{matrix} [s_k, h] \quad \text{and} \quad s_k^{a_k} h^{b_k}, & 0 \le k \le n \\ v_j h v_j^{-1} h^{-\varepsilon_j}, & 1 \le j \le g' \\ s_0 \ldots s_n V & \end{matrix} \right. \right\rangle, \tag{2}$$

where the generators and g', V are described below. Also note that if $e = 0$ then the relation $s_0^{a_0} h^{b_0}$ reduces to $s_0 = 1$, so in this case s_0 is usually omitted.

- The type $\in$ of N equals:

  o_1 if both the base surface and the total space are orientable (which forces all ε_j's to equal 1);

  o_2 if the base surface is orientable and the total space is non-orientable, hence $g \ge 1$ (which forces all ε_j's to equal -1);

  n_1 if both the base surface and the total space are non-orientable (hence $g \ge 1$) and moreover, all ε_j's equal 1;

  n_2 if the base surface is non-orientable (hence $g \ge 1$) and the total space is orientable (which forces all ε_j's to equal -1);

  n_3 if both the base surface and the total space are non-orientable and moreover, all ε_j's equal -1 except $\varepsilon_1 = 1$, and $g \ge 2$;

  n_4 if both the base surface and the total space are non-orientable and moreover, all ε_j's equal -1 except $\varepsilon_1 = \varepsilon_2 = 1$, and $g \ge 3$.

  We note that these six types, in Seifert's original notation, are respectively denoted Oo, No, Nn, On, NnI, $NnII$, where the first (capital) letter refers to the orientability or non-orientability of the total space N, while the second (lower case) letter refers to the same for the base surface.

- The orientability of the base surface and its genus g determine the number g' of the generators v_j's and the word V in the last relator of $\pi_1(N)$ as follows:

  – when the base surface is orientable, $g' = 2g$ and $V = [v_1, v_2] \ldots [v_{2g-1}, v_{2g}]$;

  – when the base surface is non-orientable, $g' = g$ and $V = v_1^2 \ldots v_g^2$.

- The generator h corresponds to the generic regular fibre.
- The generators s_k for $0 \leq k \leq n$ correspond to (possibly) exceptional fibres.

Throughout this paper, we shall use the following notations (the last one S_ϕ depends on ϕ, all the previous ones only on N).

Notation 2.2. *Let N be a Seifert manifold described by a list of Seifert invariants*

$$\{e; (\in, g); (a_1, b_1), \ldots, (a_n, b_n)\}.$$

- *Denoting by a the least common multiple of the a_k's,*

$$c = \sum_{k=0}^{n} b_k (a/a_k).$$

- *The number of even a_k's will be denoted by d.*
- *We distinguish three cases:*
    - *Case 1,* $d = 0$ *and c is even;*
    - *Case 2,* $d = 0$ *and c is odd;*
    - *Case 3,* $d > 0$.
- *In Case 3, the indices k are reordered by decreasing 2-valuation $\nu_2(a_k)$. Hence the set of even a_k's will be $\{a_0, \ldots, a_{d-1}\}$ and the set of k's for which a_k has maximal 2-valuation, denoted by S_N, will be $\{0, \ldots, J-1\}$ for some $0 < J \leq d$. Note that after this reordering, in Case 3, $a_0 \neq 1$.*
- *S_ϕ will denote the set of k's for which $\phi(s_k) = 1$.*

Note that these cases are not related to the type $\in$, each of the three cases can occur with any of the six types. The next lemma will be useful in Section 3.

Lemma 2.3. *In Case 3 ($d > 0$), c has the same parity as J. Furthermore, one also has $S_\phi \subseteq \{0, \ldots, d-1\}$, $|S_\phi|$ is even, and $\phi(h) = 0$.*

Proof. With the above notational conventions, a/a_k is odd if and only if $k < J$, and for such k's, b_k is also odd since it is coprime to a_k. Hence, modulo 2, $c = \sum b_k(a/a_k) \equiv \sum_{0\le k<J} 1 = J$. The fact that $S_\phi \subseteq \{0,\ldots,d-1\}$ follows directly from the definition of d and the reordering convention in Case 3. If we take any $k \in \{0,\ldots,d-1\}$ we have a_k even and b_k odd, hence $0 = \phi(s_k^{a_k}h^{b_k}) = a_k\phi(s_k) + b_k\phi(h) = \phi(h)$. Finally, note that $\phi(V) = 0$ in both the case of orientable or non-orientable base surface, since ϕ is a homomorphism and $\mathrm{Im}(\phi) \subseteq \mathbb{Z}_2$. Then $0 = \phi(s_0 \cdots s_n V) = \phi(s_0) + \ldots \phi(s_n)$ implies $|S_\phi|$ is even. $\square$

We close this section with an abelianized version of (2), which gives a presentation of $H_1(N) = H_1(N;\mathbb{Z})$. This will also be useful for the work in Section 3.

$$H_1(N) = \left\langle \begin{matrix} s_0,\ldots,s_n \\ v_1,\ldots,v_{g'} \\ h \end{matrix} \;\middle|\; \begin{matrix} a_k s_k + b_k h, & 0 \le k \le n \\ (1-\varepsilon_j)h, & 1 \le j \le g' \\ s_0 + \ldots + s_n + V & \end{matrix} \right\rangle, \tag{3}$$

where $V = 0$ for types o_1 and o_2, and $V = 2(v_1 + \ldots + v_g)$ for the four remaining types.

3 Study of $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau) \ge 2$

As before, let $\phi\colon \pi_1(N) \twoheadrightarrow \mathbb{Z}_2$ be an epimorphism and $\xi \in H^1(N;\mathbb{Z}_2)$ the corresponding cohomology class as given in (1). By Theorem 2.1, the set of ξ's for which $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau) = 1$ is the image of the coefficient homomorphism $\rho\colon H^1(N;\mathbb{Z}) \to H^1(N;\mathbb{Z}_2)$, so our initial goal in this section is to compute $\mathrm{Im}(\rho)$ (a less direct method, leading to the same results, would be to compute the kernel of the Bockstein homomorphism $H^1(N;\mathbb{Z}_2) \to H^2(N;\mathbb{Z})$). This is done in Propositions 3.1, 3.3, and 3.4. Then, in 3.5, we determine when $\xi \notin \mathrm{Im}(\rho)$, and this is equivalent to $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau) \ge 2$.

$>$From the presentation (3) of $H_1(N)$ we shall compute $H^1(N;\mathbb{Z}_2)$ (Proposition 3.1, which will be repeated later as a small part of Theorem 4.1), and similarly compute $H^1(N;\mathbb{Z})$ (Proposition 3.3). We use the fact that $H^1(N;\mathbb{Z}_2)$ naturally identifies to the subspace of cocycles contained in $C^1(N,\mathbb{Z}_2) := hom(C_1(N),\mathbb{Z}_2)$, where $C_1(N)$ is the free abelian group with generators v_j, s_k, h. Furthermore, using the isomorphism (1), we see that the 1-cocycles are simply the 1-cochains that vanish on the abelianized relations for $\pi_1(N)$, as given in (3). We denote by $\hat{v}_j$ $(1 \leq j \leq g')$, $\hat{s}_k$ $(0 \leq k \leq n)$, $\hat{h}$, the elements of the dual basis of $C^1(N, \mathbb{Z}_2)$ corresponding respectively to v_j, s_k, and h. In Proposition 3.3, the same notations and identifications will be used, replacing $\mathbb{Z}_2$ by $\mathbb{Z}$, recalling also that $H^1(X;\mathbb{Z})$ is a free abelian group for any finite CW-complex X.

Proposition 3.1. *Let* $\alpha = \hat{h} + \sum_{k=0}^{n} b_k \hat{s}_k$ *and* $\alpha_k = \hat{s}_k + \hat{s}_0,\ 1 \leq k \leq n$. *A basis of the* $\mathbb{Z}_2$*-vector space* $H^1(N;\mathbb{Z}_2) \subseteq C^1(N, \mathbb{Z}_2)$ *is (with Notation 2.2):*

- *Case 1 :* $\{\hat{v}_1, \ldots, \hat{v}_{g'}, \alpha\}$,
- *Case 2 :* $\{\hat{v}_1, \ldots, \hat{v}_{g'}\}$,
- *Case 3 :* $\{\hat{v}_1, \ldots, \hat{v}_{g'}, \alpha_1, \ldots, \alpha_{d-1}\}$.

Proof. Consider an arbitrary element

$$u = x\hat{h} + \sum_{k=0}^{n} z_k \hat{s}_k + \sum_{j=1}^{g'} y_j \hat{v}_j \in C^1(N;\mathbb{Z}_2)$$

(with $z_k, y_j, x \in \mathbb{Z}_2$). Due to the presentation (2) of $\pi_1(N)$, $u \in H^1(N;\mathbb{Z}_2)$ if and only if the following $n+2$ equations, coming from the relations in (3), are satisfied:

$$a_k z_k + b_k x = 0, \quad k = 0, \ldots, n, \qquad \text{and} \qquad z_0 + \ldots + z_n = 0.$$

When $d = 0$ all a_k and a are odd, so $c = \sum b_k$ and this system is thus equivalent to:

$$z_k = b_k x \ (k = 0, \ldots, n) \qquad \text{and} \qquad cx = 0.$$

The elements of $H^1(N;\mathbb{Z}_2)$ are therefore the u's of the form:

$$u = x\left(\hat{h} + \sum_{k=0}^{n} b_k \hat{s}_k\right) + \sum_{j=1}^{g'} y_j \hat{v}_j,$$

with no restriction on x in Case 1 ($d = 0$ and c even), but with $x = 0$ in Case 2 ($d = 0$ and c odd).

When $d > 0$, the system is equivalent to:

$$x = 0, \qquad z_k = b_k x \ (k = d, \dots, n) \qquad \text{and} \qquad z_0 + \dots + z_n = 0,$$

which simplifies to:

$$x = z_d = \dots = z_n = 0 \qquad \text{and} \qquad z_0 = z_1 + \dots + z_{d-1}.$$

So, in Case 3, the elements of $H^1(N;\mathbb{Z}_2)$ are the u's of the form:

$$\sum_{1 \le k \le d-1} z_k(\hat{s}_k + \hat{s}_0) + \sum_{j=1}^{g'} y_j \hat{v}_j,$$

which completes the proof. □

Remark 3.2. *In future use of this proposition and the following ones it will be important to note that if the cohomology class $u \in H^1(N;\mathbb{Z}_2)$ corresponds to the epimorphism $\phi\colon \pi_1(N) \twoheadrightarrow \mathbb{Z}_2$ via the isomorphism (1), then $z_k = \phi(s_k)$, $y_j = \phi(v_j)$, and $x = \phi(h)$. In this case we also write $u = \xi_\phi = \xi$, as in Section 1.*

Proposition 3.3. *The abelian group $H^1(N;\mathbb{Z})$ is free and generated by the following elements of $C^1(N,\mathbb{Z})$:*

- *if $\in = o_2$: $\{\hat{v}_1, \dots, \hat{v}_{g'}\}$,*
- *if $\in = n_2, n_3, n_4$: $\{\hat{v}_2 - \hat{v}_1, \dots, \hat{v}_{g'} - \hat{v}_1\}$,*
- *if $\in = o_1$:*
  - *if $c = 0$: $\{\hat{v}_1, \dots, \hat{v}_{g'},\ a\hat{h} - \sum_{k=0}^{n} b_k(a/a_k)\hat{s}_k\}$,*
  - *if $c \neq 0$: $\{\hat{v}_1, \dots, \hat{v}_{g'}\}$,*
- *if $\in = n_1$:*
  - *if c is even: $\{(c/2)\hat{v}_1 + a\hat{h} - \sum_{k=0}^{n} b_k(a/a_k)\hat{s}_k,\ \hat{v}_2 - \hat{v}_1, \dots, \hat{v}_{g'} - \hat{v}_1\}$,*
  - *if c is odd: $\{c\hat{v}_1 + 2a\hat{h} - 2\sum_{k=0}^{n} b_k(a/a_k)\hat{s}_k,\ \hat{v}_2 - \hat{v}_1, \dots, \hat{v}_{g'} - \hat{v}_1\}$.*

Proof. It was noted earlier in this section that $H^1(N;\mathbb{Z})$ is free. As in the proof of Proposition 3.1, consider an arbitrary element

$$u = x\hat{h} + \sum_{k=0}^{n} z_k \hat{s}_k + \sum_{j=1}^{g'} y_j \hat{v}_j \in C^1(N;\mathbb{Z}),$$

now with $z_k, y_j, x \in \mathbb{Z}$. We obtain that $u \in H^1(N;\mathbb{Z})$ if and only if the following equations are satisfied:

$$a_k z_k + b_k x = 0, \quad k = 0, \dots, n;$$

$$(1-\varepsilon_j)x = 0, \quad j = 1, \dots, g';$$

$$\sum_{k=0}^{n} z_k = 0 \text{ if } \in= o_1, o_2; \ \sum_{j=0}^{n} z_k + 2\sum_{j=1}^{g'} y_j = 0 \text{ if } \in= n_1, n_2, n_3, n_4.$$

Let us first treat the four easiest cases. As soon as some ε_j equals -1 (i.e. $\in= o_2, n_2, n_3, n_4$), the equation involving such a ε_j implies $x = 0$, which, by the first $n+1$ equations, forces all z_k's to be also zero. The remaining last equation thus reduces to $0 = 0$ if $\in= o_2$ and to $y_1 = -\sum_{j>1} y_j$ if $\in= n_2, n_3, n_4$. This already enables us to assert that the elements of $H^1(N;\mathbb{Z})$ are the u's of the form:

- if $\in= o_2$: $\sum y_j \hat{v}_j$
- if $\in= n_2, n_3, n_4$: $\sum_{j>1} y_j(\hat{v}_j - \hat{v}_1)$.

In the two remaining cases $\in= o_1, n_1$ (where the conditions $(1-\varepsilon_j)x = 0$ are vacuous since $\varepsilon_j = 1$), first note that the first $n+1$ equations imply that each a_k divides x, hence so does a (their l.c.m.). Letting $x = ax'$, these equations may be rewritten:

$$z_k = -b_k(a/a_k)x', \quad k = 0, \dots, n.$$

The remaining last equation thus becomes:

$$cx' = 0 \qquad \text{if} \qquad \in= o_1; \qquad cx' = 2\sum y_j \qquad \text{if} \qquad \in= n_1.$$

When $\in= o_1$, this last equation forces x' (hence also the z_k's) to be 0 if and only if $c \neq 0$. Hence the elements of $H^1(N;\mathbb{Z})$ are the u's of the form:

- if $\in = o_1$ and $c \neq 0$: $\sum y_j \hat{v}_j$
- if $\in = o_1$ and $c = 0$: $x'\left(a\hat{h} - \sum_{k=0}^{n} b_k(a/a_k)\hat{s}_k\right) + \sum y_j \hat{v}_j$.

In the last remaining case ($\in = n_1$), the last equation forces x' to be even whenever c is odd, which naturally leads us to consider two subcases:

- if c is even, this equation amounts to $y_1 = (c/2)x' - \sum_{j>1} y_j$;
- if c is odd, letting $x' = 2x''$ allows rewriting the equation as $y_1 = cx'' - \sum_{j>1} y_j$.

Hence the elements of $H^1(N;\mathbb{Z})$ are the u's of the form:

- if $\in = n_1$ and c is even:

$$x'\left(a\hat{h} - \sum_{k=0}^{n} b_k(a/a_k)\hat{s}_k + (c/2)\hat{v}_1\right) + \sum_{j>1} y_j(\hat{v}_j - \hat{v}_1)$$

- if $\in = n_1$ and c is odd:

$$x''\left(2a\hat{h} - 2\sum_{k=0}^{n} b_k(a/a_k)\hat{s}_k + c\hat{v}_1\right) + \sum_{j>1} y_j(\hat{v}_j - \hat{v}_1),$$

  which completes the proof.

□

>From Theorem 2.1 and Propositions 3.1 and 3.3 we deduce:

Proposition 3.4. *With the notations of Proposition 3.1 and Notation 2.2, the subspace* $\mathrm{Im}(\rho) \subseteq H^1(N;\mathbb{Z}_2)$ *has basis:*

- *if* $\in = o_2$*:* $\{\hat{v}_1, \ldots, \hat{v}_{g'}\}$,
- *if* $\in = n_2, n_3, n_4$*:* $\{\hat{v}_2 + \hat{v}_1, \ldots, \hat{v}_{g'} + \hat{v}_1\}$,
- *if* $\in = o_1$*: if* $c \neq 0$ *then* $\{\hat{v}_1, \ldots, \hat{v}_{g'}\}$,
  - *if* $c = 0$ *and* $d = 0$ *then* $\{\hat{v}_1, \ldots, \hat{v}_{g'}, \alpha\}$,
  - *if* $c = 0$ *and* $d > 0$ *then* $\{\hat{v}_1, \ldots, \hat{v}_{g'}, \sum_{1 \leq k \leq J-1} \alpha_k\}$,
- *if* $\in = n_1$*:*

– *if c is odd:* $\{\hat{v}_1, \ldots, \hat{v}_{g'}\}$,
– *when c is even and $d = 0$:* $\{\hat{v}_2 + \hat{v}_1, \ldots, \hat{v}_{g'} + \hat{v}_1, (c/2)\hat{v}_1 + \alpha\}$,
– *if c is even and $d > 0$:* $\{\hat{v}_2 + \hat{v}_1, \ldots, \hat{v}_{g'} + \hat{v}_1, (c/2)\hat{v}_1 + \sum_{1 \le k \le J-1} \alpha_k\}$.

Proof. Most of this statement follows immediately from Propositions 3.1 and 3.3; we shall address the only non-obvious parts which are the two possibilities ($\in = o_1, c = 0$) and ($\in = n_1, c$ even). Note that in these two cases, Case 2 ($d = 0, c$ odd) does not occur.

If $\in = o_1$ and $c = 0$, we must compute the image in $H^1(N; \mathbb{Z}_2)$ of $u := a\hat{h} - \sum_{k=0}^{n} b_k(a/a_k)\hat{s}_k \in H^1(N; \mathbb{Z})$, in terms of the generators of $H^1(N; \mathbb{Z}_2)$.

- If $d = 0$ (Case 1, a, a_k are all odd): $\rho(u) = \hat{h} + \sum_{k=0}^{n} b_k\hat{s}_k = \alpha$.
- If $d > 0$ (Case 3, a/a_k is odd only for $0 \le k \le J-1$, a is even, and $b_0, \ldots, b_{d-1}$ are odd):

$$\rho(u) = 0 \cdot \hat{h} + \sum_{k=0}^{n} b_k(a/a_k)\hat{s}_k = \sum_{k=0}^{J-1} b_k\hat{s}_k = \sum_{k=0}^{J-1} \hat{s}_k.$$

  By Lemma 2.3 J is even, so we may rewrite this sum as $\sum_{k=1}^{J-1}(\hat{s}_k + \hat{s}_0) = \sum_{k=0}^{J-1} \alpha_k$, as desired.

- If $\in = n_1$ and c is even, the proofs in the two cases (Case 1 and Case 3) are identical to the corresponding previous two cases for $\in = o_1$, except that $(c/2)\hat{v}_1$ is added to u and hence also to $\rho(u)$.

□

We now take into account Remark 3.2, Notation 2.2, and Proposition 3.4 (in its negated form), to prove the main theorem of this section.

Theorem 3.5. *One has* $\mathrm{ind}_{\mathbb{Z}_2}(M_\phi, \tau_\phi) \in \{2, 3\}$ *in exactly the following cases:*

- *Either $\in = o_1$ and $c \neq 0$, or $\in = n_1$ and c is odd, or $\in = o_2$, and in addition* $\{\phi(h), \phi(s_0), \ldots, \phi(s_n)\} \neq \{0\}$,
- *Either $\in = n_2$ or $\in = n_3$ or $\in = n_4$, and in addition* $\{\phi(\sum v_j), \phi(h), \phi(s_0), \ldots, \phi(s_n\} \neq \{0\}$,

- $\in= o_1, c = 0, d > 0$ *and* $S_\phi \neq \emptyset, S_N$
- $\in= n_1$, *c is even and:*
  - *if* $d = 0$*:* $\sum_{j=1}^{g'} \phi(v_j) \neq (c/2)\phi(h)$
  - *if* $d > 0$*: either* $S_\phi \neq \emptyset, S_N$*, or* $S_\phi = \emptyset$ *and* $\sum_{j=1}^{g'} \phi(v_j) \neq 0$*, or* $S_\phi = S_N$ *and* $\sum_{j=1}^{g'} \phi(v_j) \neq (c/2)$*.*

Proof. Writing as usual $\xi = \xi_\phi \in H^1(N; \mathbb{Z}_2)$, the condition given in Theorem 2.1(1) tells us that $\mathrm{ind}_{\mathbb{Z}_2}(M_\phi, \tau_\phi) \in \{2, 3\}$ if and only if $\xi \notin \mathrm{Im}(\rho)$. Now Proposition 3.4 identifies $\mathrm{Im}(\rho)$, so in each case we simply have to negate the conditions given in Proposition 3.4.

- When either $\in= o_1$ and $c \neq 0$, or $\in= n_1$ and c is odd, or $\in= o_2$, $\mathrm{Im}(\rho) = \langle \hat{v}_1, \ldots, \hat{v}_{g'} \rangle$. Therefore $\xi = x\hat{h} + \sum_{k=0}^{n} z_k \hat{s}_k + \sum_{j=1}^{g'} y_j \hat{v}_j \notin \mathrm{Im}(\rho)$ if and only if some z_k or x is non-zero, which is identical to the given condition (see Remark 3.2).
- When either $\in= n_2$ or $\in= n_3$ or $\in= n_4$, $\mathrm{Im}(\rho) = \langle \hat{v}_2 + \hat{v}_1, \ldots, \hat{v}_{g'} + \hat{v}_1 \rangle$. Therefore $\xi \in \mathrm{Im}(\rho)$ if and only if $\xi = \sum_{j=2}^{g'} y_j(\hat{v}_j + \hat{v}_1)$, or equivalently $\xi = \sum_{j=1}^{g'} y_j \hat{v}_j$ with $\sum_{j=1}^{g'} y_j = 0$. So $\xi \notin \mathrm{Im}(\rho)$ if and only if some $x_k \neq 0$ or $x \neq 0$ or $\sum_{j=1}^{g'} y_j \neq 0$, which is identical to the given condition.
- When $\in= o_1, c = 0, d = 0$, we see from Propositions 3.4 and 3.1 (Case 1) that ρ is surjective. So $\xi \in \mathrm{Im}(\rho)$, i.e. $\mathrm{ind}_{\mathbb{Z}_2}(M, \phi) = 1$, and hence this case does not appear on the list in Theorem 3.5.
- When $\in= o_1, c = 0, d > 0$, we have Case 3 so Lemma 2.3 applies, and we shall use it several times here. In particular we will use $0 = c \equiv J \pmod 2$ and $x = \phi(h) = 0$ without further mention. Here $\mathrm{Im}(\rho) = \langle \hat{v}_1, \ldots, \hat{v}_{g'}, \sum_{k=1}^{J-1} \alpha_k \rangle$, hence $\xi \in \mathrm{Im}(\rho)$ if and only if, for some $y_j, z \in \mathbb{Z}_2$,

$$\xi = \sum_{j=1}^{g'} y_j \hat{v}_j + z \sum_{k=1}^{J-1} (\hat{s}_k + \hat{s}_0) = \sum_{j=1}^{g'} \hat{v}_j + \sum_{k=0}^{J-1} z \hat{s}_k.$$

  Since, as already noted, $x = 0$, we deduce $\xi = \sum_{j=1}^{g'} y_j \hat{v}_j + \sum_{k=0}^{J-1} z \hat{s}_k \notin \mathrm{Im}(\rho)$ if and only if either $\phi(\hat{s}_k) = z_k \neq 0$ for some $k \geq J$, or $z_J = \ldots = z_n = 0$ and $\{\phi(\hat{s}_0), \ldots, \varphi(\hat{s}_{J-1})\} = \{z_0, \ldots, z_{J-1}\} = \{0,1\}$ (i.e. $\varphi(\hat{s}_0), \ldots, \phi(\hat{s}_{J-1})$ are not all equal). These conditions are easily seen, recalling Notation 2.2, to be equivalent to $S_\phi \neq \emptyset$, S_N, as stated.

- When $\in = n_1$, c even, and $d = 0$, we have Case 1 so $z_k = b_k x$, $0 \leq k \leq n$, as seen in the proof of Proposition 3.1. Here $\mathrm{Im}(\rho) = \langle \hat{v}_2 + \hat{v}_1, \ldots, \hat{v}_{g'} + \hat{v}_1, (c/2)\hat{v}_1 + \alpha \rangle$. Noting that $\alpha(h) = x$ and $\hat{v}_1(h) = 0$, this gives that $\xi \in \mathrm{Im}(\rho)$ if and only if

$$\xi = \sum_{j=2}^{g'} y_j(\hat{v}_j + \hat{v}_1) + x[(c/2)\hat{v}_1 + \alpha] = \sum_{j=1}^{g'} y_j \hat{v}_j + x\alpha$$

$$= \sum_{j=1}^{g'} y_j \hat{v}_j + x\hat{h} + \sum_{k=0}^{n} z_k \hat{s}_k,$$

where $y_1 = x(c/2) + y_2 + \ldots y_{g'}$, or equivalently $\sum_{j=1}^{g'} y_j = x(c/2)$. It follows that $\xi \notin \mathrm{Im}(\rho)$ if and only if $\sum_{j=1}^{g'} y_j \neq (c/2)x$, and this is the same as the stated condition.

- When $\in = n_1, c$ even, and $d > 0$ we again have Case 3 so as in the previous Case 3, J is even and $x = 0$. Now

$$\mathrm{Im}(\rho) = \langle \hat{v}_2 + \hat{v}_1, \ldots, \hat{v}_{g'} + \hat{v}_1, (c/2)\hat{v}_1 + \sum_{k=1}^{J-1} \hat{s}_k + \hat{s}_0 \rangle.$$

Then $\xi \in \mathrm{Im}(\rho)$ if and only if $\xi = \sum_{j=2}^{g'} y_j(\hat{v}_j + \hat{v}_1) + tc_1\hat{v}_1 + t\sum_{k=0}^{n} \hat{s}_k = \sum_{j=1}^{g'} y_j \hat{v}_j + t\sum_{k=0}^{n} \hat{s}_k$, where $y_1 = y_2 + \ldots + y_{g'} + t(c/2)$, or equivalently $\sum_{j=1}^{g'} y_j = t(c/2)$. It follows that $\xi \notin \mathrm{Im}(\rho)$ if and only if either $S_\phi \not\subseteq S_N$, or $S_\phi \subseteq S_N$ and $\sum_{j=1}^{g'} \phi(v_j) \neq (c/2)\phi(s_k)$ for at least one k, $1 \leq k \leq J-1$. Again, these conditions are easily seen to be equivalent to the stated conditions in this case.

$\square$

4 Study of $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau)=3$

According to 2.1(ii), one has $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau)=3$ if and only if $\xi^3\neq 0$. We therefore begin this section by stating known results (cf. [1], [2], [3], [4]) for the $\mathbb{Z}_2$-cohomology ring of a Seifert manifold N. For $H^1(N;\mathbb{Z}_2)$ this necessarily overlaps with some of the computations done in Section 3, and the notations used in Section 3 are consistent with those in the references (where we now will write $\hat{v}_j=\theta_j$). For types $\in= o_1, o_2$ what we now call $\theta_1,\theta_2,\theta_3,\theta_4,\dots$ correspond respectively to $\theta_1,\theta_1',\theta_2,\theta_2',\dots$ in [4], while the notation is identical for the remaining four types. As far as the cup products it suffices to list just the non-zero products in positive dimensions, on the generators, also taking account that $xy=yx$ in $H^*(\_\_\,;\,\mathbb{Z}_2)$.

Theorem 4.1. *Let N be any Seifert manifold described by a list of Seifert invariants*

$$\{e;(\in,g);(a_1,b_1),\dots,(a_n,b_n)\},$$

the type $\in$ being o_1,o_2,n_1,n_2,n_3,n_4.

Using Notation 2.2, the cohomology groups $H^*(N;\mathbb{Z}_2)$ are: $H^0=\mathbb{Z}_2\{1\}$ (the unit for the cup-product), $H^3=\mathbb{Z}_2\{\gamma\}$, and (with $1\leq j\leq g'=2g$ for the types o_1 and o_2, and $1\leq j\leq g'=g$ for the other types):

- *Case 1 (if $d=0$ and c is even): $H^1=\mathbb{Z}_2\{\theta_1,\dots\theta_{g'},\ \alpha=\hat{h}+\sum_{k=0}^n b_k\hat{s}_k\}$, $H^2=\mathbb{Z}_2\{\varphi_1,\dots,\varphi_{g'},\beta\}$.*

- *Case 2 (if $d=0$ and c is odd): $H^1=\mathbb{Z}_2\{\theta_1,\dots,\theta_{g'}\}$, $H^2=\mathbb{Z}_2\{\varphi_1,\dots,\varphi_{g'}\}$,*

- *Case 3 (if $d>0$): $H^1=\mathbb{Z}_2\{\theta_1,\dots\theta_{g'},\alpha_1,\dots\alpha_{d-1}\}$,*
  $H^2=\mathbb{Z}_2\{\varphi_1,\dots,\varphi_{g'},\beta_1,\dots,\beta_{d-1}\}$.

The non-trivial cup-products, on the generators of $H^1\otimes H^1$ and $H^1\otimes H^2$, are:

- *In all three Cases, for the types o_1 and o_2, $\theta_{2i-1}\varphi_{2i}=\theta_{2i}\varphi_{2i-1}=\gamma$, while for the other types $\theta_j\varphi_j=\gamma$.*

- *in Case 1, for the types o_1 and o_2, $\theta_{2i-1}\theta_{2i}=\beta$, while for the other types $\theta_j^2=\beta$.*

– *in Case 1,* $\theta_j\alpha = \varphi_j$, $\alpha\beta = \gamma$, $\alpha\varphi_j = \gamma$ *when* $\varepsilon_j = -1$ *(as specified in Section 2 for each of the types), and*

$$\alpha^2 = (c/2)\beta + \sum_{\varepsilon_j=-1} \varphi_j.$$

– *in Case 3 (i.e.* $d > 0$*),* $\alpha_k\beta_k = \gamma$, $k > 0$, *and, for* $k, l > 0$,

$$\alpha_k\alpha_\ell = \frac{a_0}{2}\beta_0 + \delta_{k,\ell}\frac{a_k}{2}\beta_k,$$

where β_0 *denotes* $\sum_{1\leq k\leq d-1} \beta_k$.

>From this theorem, we deduce:

Proposition 4.2. *With the same notations, let* $\xi = \xi_\phi \in H^1(N; \mathbb{Z}_2)$.

- *In Case 1,*

$$\xi^3 = \begin{cases} \phi(h)(c/2)\cdot\gamma \text{ when } \in = o_1, \\ \phi(h)((c/2) + \sum\phi(v_i))\cdot\gamma \text{ when } \in = o_2, n_1, \\ \phi(h)((c/2) + g)\cdot\gamma \text{ when } \in = n_2, \\ \phi(h)((c/2) + \phi(v_1) + g - 1)\cdot\gamma \text{ when } \in = n_3, \\ \phi(h)((c/2) + \phi(v_1) + \phi(v_2) + g)\cdot\gamma \text{ when } \in = n_4. \end{cases}$$

- *In Case 2,* $\xi^3 = 0$.
- *In Case 3,* $\xi^3 = (\sum\phi(s_k)(a_k/2))\cdot\gamma$.

Proof.

Case 1. Let $\xi = x\cdot\alpha + \sum y_j\cdot\theta_j$ with $x = \phi(h)$ and $y_j = \phi(v_j)$, then

$$\xi^2 = x\cdot\alpha^2 + \sum y_j\cdot\theta_j^2 = x((c/2)\cdot\beta + \sum_{\varepsilon_j=-1}\varphi_j) + y\cdot\beta,$$

with $y = 0$ when $\in = o_1, o_2$, $y = \sum y_j$ when $\in = n_1, n_2, n_3, n_4$, and $\sum_{\varepsilon_j=-1}\varphi_j = 0$ for types o_1, n_1. For the various types, this now gives :

– when $\in = o_1$, $\xi^3 = (x\alpha + \sum_{j=1}^{2g} y_j\theta_j)\cup x(c/2)\beta = x(c/2)\cdot\gamma$.

– when $\in = o_2$,

$$\begin{aligned}
\xi^3 &= (x\alpha + \sum_{j=1}^{2g} y_j\theta_j) \cup x((c/2)\beta + \sum_{j=1}^{2g} \varphi_j) \\
&= x((c/2) + 2g + \sum_{j=1}^{2g} y_j) \cdot \gamma = x((c/2) + \sum_{j=1}^{2g} y_j) \cdot \gamma.
\end{aligned}$$

– when $\in = n_1, n_2, n_3, n_4$,

$$\begin{aligned}
\xi^3 &= (x\alpha + \sum_{j=1}^{g} y_j\theta_j) \cup \left((x(c/2) + y)\beta + x \sum_{\varepsilon_j=-1} \varphi_j \right) \\
&= x \left((c/2) + y + \#\{j \mid \varepsilon_j = -1\} + \sum_{\varepsilon_j=-1} y_j \right) \cdot \gamma \\
&= x \left((c/2) + \sum_{\varepsilon_j=1} y_j + \#\{j \mid \varepsilon_j = -1\} \right) \cdot \gamma.
\end{aligned}$$

Case 2. $\xi = \sum y_j\theta_j$, hence $\xi^2 = \sum y_j\theta_j^2 = 0$ and $\xi^3 = 0$.

Case 3. Letting $z_k = \phi(s_k)$, recall from the proof of Proposition 3.1 that $z_k = 0$ for $k \geq d$, $z_0 = \sum_{k>0} z_k$, and $\xi = \sum_{1\leq k\leq d-1} z_k\alpha_k + \sum y_j\theta_j$, hence

$$\begin{aligned}
\xi^2 &= \sum_{1\leq k\leq d-1} z_k\alpha_k^2 + \sum y_j\theta_j^2 \\
&= \sum_{1\leq k\leq d-1} z_k(\frac{a_0}{2}\beta_0 + \frac{a_k}{2}\beta_k) + 0 \\
&= \frac{a_0}{2}(\sum_{1\leq k\leq d-1} z_k)\beta_0 + \sum_{1\leq k\leq d-1} z_k\frac{a_k}{2}\beta_k \\
&= \frac{a_0}{2}z_0\beta_0 + \sum_{1\leq k\leq d-1} z_k\frac{a_k}{2}\beta_k \\
&= \sum_{0\leq k\leq d-1} z_k\frac{a_k}{2}\beta_k
\end{aligned}$$

and

$$\begin{aligned}
\xi^3 &= \Big(\sum_{1\le k\le d-1} z_k\alpha_k + \sum y_j\theta_j\Big) \cup \sum_{0\le k\le d-1} z_k\frac{a_k}{2}\beta_k \\
&= z_0\frac{a_0}{2}\Big(\sum_{1\le k\le d-1} z_k\alpha_k\Big)\cup\beta_0 + \sum_{1\le k\le d-1} z_k\frac{a_k}{2}\cdot\gamma \\
&= z_0\frac{a_0}{2}\Big(\sum_{1\le k\le d-1} z_k\Big)\gamma + \sum_{1\le k\le d-1} z_k\frac{a_k}{2}\cdot\gamma \\
&= z_0\frac{a_0}{2}\gamma + \sum_{1\le k\le d-1} z_k\frac{a_k}{2}\cdot\gamma \\
&= \sum_{0\le k\le d-1} z_k\frac{a_k}{2}\cdot\gamma.
\end{aligned}$$

□

Using Proposition 4.2, we conclude:

Theorem 4.3. *One has* $\mathrm{ind}_{\mathbb{Z}_2}(M_\phi,\tau_\phi)=3$ *if and only if*

- *either N satisfies Case 3 (i.e. $d>0$) and $\sum_{\phi(s_k)=1} a_k$ is not a multiple of 4,*
- *or N satisfies Case 1 (i.e. $d=0$ and c is even), and $\phi(h)=1$, and the following element of $\mathbb{Z}_2$ is nonzero:*
  - *when $\in = o_1$: $c/2$*
  - *when $\in = o_2, n_1$: $(c/2)+\sum\phi(v_j)$*
  - *when $\in = n_2$: $(c/2)+g$*
  - *when $\in = n_3$: $(c/2)+\phi(v_1)+g-1$*
  - *when $\in = n_4$: $(c/2)+\phi(v_1)+\phi(v_2)+g$.*

5 Remarks and examples

In this section we give a brief discussion of the class ξ^2 and several examples. The first few examples tend to involve relatively simple Seifert manifolds for which the full machinery of the previous sections is not strictly needed. The final two examples are more involved and the full machinery will be necessary. These examples cover three of the six possible Seifert manifold types, namely $\in = o_1, n_1, n_3$, as well as various Euler numbers e and genus g'.

Proposition 5.1. *(a) If* $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau) = 1$, *then* $\xi^2 = 0$.
(b) If $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau) = 3$, *then* $\xi^2 \neq 0$.

Proof. (a) Consider the Bockstein homomorphisms $B\colon H^1(N;\mathbb{Z}_2) \to H^2(N;\mathbb{Z})$ and $\beta = Sq^1\colon H^1(N;\mathbb{Z}_2) \to H^2(N;\mathbb{Z}_2)$, and recall that under the coefficient homomorphism $\rho'\colon H^2(N;\mathbb{Z}) \to H^2(N;\mathbb{Z}_2)$ one has $\rho' \circ B = \beta$. From Theorem 2.1(i) we know $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau) = 1$ if and only if $\xi \in \mathrm{Im}(\rho)$. Since $\mathrm{Im}(\rho) = \mathrm{Ker}(B)$, the condition is equivalent to $B(\xi) = 0$. And this implies $0 = Sq^1(\xi) = \xi^2$.

(b) This is immediate from Theorem 2.1(ii).

□

Based on 5.1 (a), it is interesting to have examples where $\xi^2 = 0$ and where the $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau)$ could equal 1 or equal 2. In fact such examples are already considered in [7], Section 5, and we will recall them here.

Example 5.2.

(a) Let $N = L(4,1)$, and $\xi \in H^1(N;\mathbb{Z}_2) \approx \mathbb{Z}_2$ be the generator. Then $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau) = 2$ and $\xi^2 = 0$.

(b) Let $N = S^1 \times V$, V being is any closed surface, and $\xi = \pi^*(u)$, where $\pi\colon N \twoheadrightarrow S^1$ is the projection and u generates $H^1(S^1;\mathbb{Z}_2)$. Then $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau) = 1$ and $\xi^2 = 0$.

(c) As a special case of (b) let $N = S^1 \times \mathbb{R}P^2$, then $H^1(N;\mathbb{Z}_2)$ has the generator u as in (b), and the additional generator x corresponding to the (pull-back) of the generator of $H^1(\mathbb{R}P^2;\mathbb{Z}_2)$. Now, in addition to $\xi = u$ as in (b), we have two further possible choices $\xi = v$ or $\xi = u + v$. For each of these latter two choices we have $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau) = 2$ since $\xi^2 = v^2 \neq 0$ and $\xi^3 = 0$.

Of course, the conclusions in Example 5.2 as well as the following Example 5.3 also follow easily from our main theorems. As an illustration, in 5.2(a) we have $L(4,1) = \{4; (o_1, 0)\}$ (cf. [15] 5.4(i)). Here $a_0 = 1$, $b_0 = 4$, whence $d = 0$, $c = 4$, and this implies we are in Case 1. By Theorem 4.1 the only non-zero element in $H^1(N;\mathbb{Z}_2)$ is α, hence $\xi = \alpha$. Again by Theorem 4.1 we have $\alpha^2 = (c/2)\beta = 0$. Now applying Theorem 3.5 (first case) and Theorem 4.3 (second case), we obtain that $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau) = 2$.

We also remark that in 5.2(b) and 5.2(c) one has $e = 0$, and the type is o_1 if V is orientable, n_1 if V is non-orientable.

Our next example illustrates to some extent the delicacy of the Borsuk-Ulam situation. The example shows that one can have two double covers of a Seifert manifold N by the same Seifert manifold M but with different $\mathbb{Z}_2$-indices for (M, τ). Indeed the example already arises at the level of surface topology.

Example 5.3.

Let $N = \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$, one has $\pi_1(N) = \langle v_1, v_2, v_3 | v_1^2 v_2^2 v_3^2 \rangle$, $H^1(N; \mathbb{Z}_2) \approx \mathbb{Z}_2^3$ with generators $\theta_1, \theta_2, \theta_3$ and $H^2(N; \mathbb{Z}_2) \approx \mathbb{Z}_2$ with generator β. Furthermore $\theta_i^2 = \beta$ whereas $\theta_i \theta_j = 0$, $i \neq j$ (cf. [8] Section 3.2). The characteristic class $\xi_1 = \theta_2 + \theta_3$ corresponds to the homomorphism $\phi_1 \colon \pi_1(N) \twoheadrightarrow \mathbb{Z}_2$ given by $\phi_1(v_1) = 0, \phi_1(v_2) = \phi_1(v_3) = 1$. Similarly the characteristic class $\xi_2 = \theta_1$ corresponds to $\phi_2 \colon \pi_1(N) \twoheadrightarrow \mathbb{Z}_2$ with $\phi_2(v_1) = 1, \phi_2(v_2) = \phi_2(v_3) = 0$. Using Proposition 4.2 of [7] we obtain at once that $\mathrm{ind}_{\mathbb{Z}_2}$ is 1 for ξ_1 and 2 for ξ_2 (this corresponds to $\xi_1^2 = 0$, $\xi_2^2 \neq 0$). The surface M that is the double cover of N must have Euler characteristic $\chi(M) = 2\chi(N) = -2$. Since it is not hard to see that in both cases M is non-orientable, it follows that in both cases $M = \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$.

By simply taking the product of M and N with S^1, we obtain similar examples with Seifert manifolds (where we take $\phi_i(h) = 0$). Indeed, writing $N_1 = N \times S^1$, we have that N_1 has $\in = n_1$, $g' = 3$, and no exceptional fibres whence $d = c = 0$. From 3.5, final case, we see that $\phi_1(v_1) + \phi_1(v_2) + \phi_1(v_3) = 0$ implies the $\mathbb{Z}_2$-index for ϕ_1 equals 1, whereas $\phi_2(v_1) + \phi_2(v_2) + \phi_2(v_3) = 1$ implies the $\mathbb{Z}_2$ index for ϕ_2 is 2 or 3. Since $\zeta^3 = 0$ for any $\zeta \in H^1(N_1; \mathbb{Z}_2)$, the $\mathbb{Z}_2$-index of ϕ_2 must be 2.

It should be noted, as was already done in Seifert's original paper [19], that the same 3-manifold (even S^3) can often be fibred in different ways, i.e. the Seifert "invariants" are not always true invariants in the sense that they may not be unique. However the cohomology ring with any coefficients, and fundamental group, are of course true invariants, and the determination of the $\mathbb{Z}_2$-index is based upon these. We conclude with a couple of deeper examples for which the techniques of Sections 3 and 4 must be utilized to answer the Borsuk-Ulam question.

Example 5.4.

Let N be the Seifert manifold given by the following Seifert invariants:

$$N = \{0, (n_3, 2); (9, 4), (5, 2), (7, 2)\}.$$

Then, a presentation of $\pi_1(N)$ is:

$$\pi_1(N) = \left\langle \begin{matrix} s_1, s_2, s_3 \\ v_1, v_2 \\ h \end{matrix} \left| \begin{matrix} [s_k, h] \quad (k = 1, 2, 3) \\ [v_1, h], \quad v_2 h v_2^{-1} h \\ s_1^9 h^4, \ s_2^5 h^2, s_3^7 h^2, \ s_1 s_2 s_3 v_1^2 v_2^2 \end{matrix} \right. \right\rangle.$$

Note that $d = 0$ (since $9, 5, 7$ are odd) and c is even (since $4, 2, 2$ are even), hence we are in Case 1 of Notation 2.2. The following table shows the values of all possible non-zero ϕ's on the generators of $\pi_1(N)$, as well as the corresponding cohomology class $\xi \in H^1(N; \mathbb{Z}_2)$ under the isomorphism (1). Also recall that here, by Theorem 4.1 (or Proposition 3.1), $H^1(N; \mathbb{Z}_2)$ has generators $\alpha, \theta_1, \theta_2$ with $\alpha = \hat{h} + 4\hat{s}_1 + 2\hat{s}_2 + 2\hat{s}_3 = \hat{h}$, and finally that $\in = n_3$ implies all $\phi(s_j) = 0$. The final column in the table gives the $\mathbb{Z}_2$-index, in each case, of $(M_i, \tau_i) := (M_{\xi_i}, \tau_{\xi_i})$. The proofs for the data in the table are given in Proposition 5.5 below.

ϕ_i	s_1	s_2	s_3	h	v_1	v_2	ξ_i	$\mathrm{ind}_{\mathbb{Z}_2}(M_i, \tau_i)$
ϕ_1	0	0	0	1	0	0	α	3
ϕ_2	0	0	0	1	1	0	$\alpha + \theta_1$	2
ϕ_3	0	0	0	1	0	1	$\alpha + \theta_2$	3
ϕ_4	0	0	0	1	1	1	$\alpha + \theta_1 + \theta_2$	2
ϕ_5	0	0	0	0	1	0	θ_1	2
ϕ_6	0	0	0	0	0	1	θ_2	2
ϕ_7	0	0	0	0	1	1	$\theta_1 + \theta_2$	1

Proposition 5.5.

- *For $\xi = \xi_7$ one has* $\mathrm{ind}_{\mathbb{Z}_2}(M_i, \tau_i) = 1$.
- *For $\xi = \xi_2, \xi_4, \xi_5, \xi_6$ one has* $\mathrm{ind}_{\mathbb{Z}_2}(M_i, \tau_i) = 2$.
- *For $\xi = \xi_1, \xi_3$ one has* $\mathrm{ind}_{\mathbb{Z}_2}(M_i, \tau_i) = 3$.

Proof. By Theorem 3.5, $\mathrm{ind}_{\mathbb{Z}_2}(M_i, \tau_i) = 1$ if and only if $\phi(h) = \phi(v_1 + v_2) = 0$, i.e. $\phi = \phi_7$. Moreover, N is in Case 1, $\in = n_3$, c is a multiple of 4 and $g = 2$ hence, by Theorem 4.3, $\mathrm{ind}_{\mathbb{Z}_2}(M_i, \tau_i) = 3$ if and only if $\phi(h) = \phi(v_1) + 1 = 1$, i.e. $\phi = \phi_1, \phi_3$. $\square$

Our concluding example has (in contrast to the previous examples) non-zero Euler number, arbitrary genus $g \geq 0$, and a relatively large number (seven) of singular fibres.

Example 5.6.

Let $N_g,\ g \geq 0$, be the Seifert manifold given by the Seifert invariants

$$\{-2; (o_1, g); (16,5), (16,1), (16,1), (16,1), (2,1), (3,2), (3,1)\}.$$

With the conventions given in Notation 2.2, a presentation of $\pi_1(N)$ is (note that according to these conventions the singular fibres are reordered so that s_0 corresponds to $(16,5)$, s_1 to $(16,1)\ldots$, s_6 to $(3,1)$, and s_7 to $(1,e) = (1,-2)$:

$$\pi_1(N) = \left\langle \begin{matrix} s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7 \\ v_1, v_2, \ldots, v_{2g-1}, v_{2g} \\ h \end{matrix} \;\middle|\; \begin{matrix} [s_k, h] \quad \text{and} \quad s_k^{a_k} h^{b_k}, \quad 0 \leq k \leq 7 \\ [v_j, h], \quad 1 \leq j \leq 2g \\ s_0 \cdots s_7 [v_1, v_2] \cdots [v_{2g-1}, v_{2g}] \end{matrix} \right\rangle.$$

One easily checks that here $a = 48,\ c = 0,\ d = 5,\ J = 4$, whence $S_N = \{0,1,2,3\}$ and we are in Case 3 of Notation 2.2. As usual, ϕ denotes any surjective homomorphism $\phi\colon \pi_1(N_g) \twoheadrightarrow \mathbb{Z}_2$ and τ the corresponding involution of the double cover M arising from ϕ. It is also readily seen that $\phi(h) = \phi(s_5) = \phi(s_6) = \phi(s_7)$ and $\phi(s_0) + \phi(s_1) + \phi(s_2) + \phi(s_3) + \phi(s_4) = 0$ are necessary conditions for ϕ to be a homomorphism.

Proposition 5.7.

- $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau) = 1$ *iff either* $S_\phi = \emptyset$ *(in which case* $\phi(s_k) = 0,\ 0 \leq k \leq 7,\ g \geq 1$, *and* $\phi(v_j) = 1$ *for at least one* j*), or* $S_\phi = S_N$ *(in which case* $\phi(s_0) = \phi(s_1) = \phi(s_2) = \phi(s_3) = 1$*).*
- $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau) = 3$ *iff* $\phi(s_4) = 1$ *(whence also* $\phi(s_0) + \phi(s_1) + \phi(s_2) + \phi(s_3) = 1$*).*
- *In all remaining cases* $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau) = 2$.

Proof. By Theorem 3.5 we have $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau) > 1$ if and only if $d > 0$ and $S_\phi \neq \emptyset, S_N$. Since here $d = 5$, the negation of the previous sentence gives the first statement of the proposition.

By Theorem 4.3 we have $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau) = 3$ if and only if $d > 0$ (which is the case) and $\sum\{a_k : k \in S_\phi\}$ is not divisible by 4. We have already observed that $S_\phi \subseteq \{0,1,2,3,4\}$ and furthermore $a_0 = a_1 = a_2 = a_3 = 16$, hence $\sum\{a_k : k \in S_\phi\} \equiv 2 \cdot \phi(s_4) \pmod 4$, and this gives the second statement of the proposition. The third and final statement follows by default.

□

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Time scale version of the Ważewski retract method

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In the paper there are discussed approaches to the Ważewski retract method on time scales. In particular there is presented planar case without a restrictive assumption that the whole boundary of a set of constraints, where we look for solutions, is a set of egress points. One example illustrating the main theorem is presented.

Introduction

In 1947 Tadeusz Ważewski (see [1]) gave a simple but excellent topological principle, now called the Ważewski retract method, which has been used by many authors to prove the existence of solutions of a given differential equation which remain in a prescribed set of constraints. In particular, the method helps to find bounded solutions in several differential problems. It generalizes the direct method of Lyapunov and is based on examining so-called 'egress' and 'strict egress' points on a boundary of the set of constraints. It is worth noting that the set does not need to be an attractor or repellor. It is sufficient to check that the set of egress points, which is usually assumed to be equal to the set of strict egress points, is not a retract or, more generally, strong deformation retract of the whole set. This topological principle became a base and a motivation for a construction of a very well known and useful topological invariant, the Conley index (see, e.g., [2] for a comparison of these two topological tools).

The Ważewski retract method was generalized and adopted to: differential inclusions (see, e.g., [2] or [3] and references therein), difference equations (e.g. [4, 5]) or, recently, dynamic equations on time scales ([6, 7, 8]). This last area of research has been intensively developed since 90's as a unification and generalization of the theory of difference equations and differential equations, and has found applications in many mathematical models in biology and physics, where discrete and continuous dynamics have to be studied simultaneously. Moreover, various impulsive differential problems can be transformed to dynamic equations on time scales.

While several results on dynamic equations on time scales are just simple transformations of continuous or discrete analogs, the ones concerning qualitative theory are not. The results on the Ważewski topological principle for dynamic equations on time scales are still not satisfactory. In fact, the only cases explored enough are the ones where the set of constraints is negatively invariant (see [6, 7]).

When we drop the above simplification, we meet several essential problems. The main of them is to construct a retraction, which has to be a continuous map, from an initial section Ω_{t_0} of the tube of constraints onto the t_0-section E_{t_0} of the set of egress points. We need a deep geometrical study to overcome this problems. The Shöenflies theorem, a convexity and strict convexity play an important role in proofs of non-whole boundary case (see [8]).

The paper is organized as follows. In section 2 we recall some information on the calculus on time scales that will be useful in the sequel. Section 3 shows topological ideas contained in [6] and [7]. Section 4 presents results from [8], where the positive or negative invariance as well as a repulsivity of the set is not assumed anymore. One transparent example is given to illustrate the results.

2 Preliminaries

2.1 Basics of a calculus on time scales

The interested reader can consult [9, 10] to get a complete introduction or to find proofs of statements of this section.

A time scale is any closed subset of the set $\mathbb{R}$ of real numbers and we denote it by $\mathbb{T}$.

Basic functions describing $\mathbb{T}$ are jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ i $\mu : \mathbb{T} \to \mathbb{R}$, defined as follows:

- $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ (forward jump operator)
- $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$ (backward jump operator)
- $\mu(t) = \sigma(t) - t$ (graininess function)

where we assume: $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$.

Proposition 2.1 (Induction Principle)**.** *Let $t_0 \in \mathbb{T}$ and assume that $\{S(t) : t \in [t_0, \infty) \cap \mathbb{T}\}$ is a family of statements satisfying:*

- *The statemnt $S(t_0)$ is true.*
- *If $t \in [t_0, \infty) \cap \mathbb{T}$ is right-scattered and $S(t)$ is true for all $s \in [t_0, t) \cap \mathbb{T}$, then $S(\sigma(t))$ is also true.*
- *If $t \in [t_0, \infty) \cap \mathbb{T}$ is right-dense and $S(t)$ is true, then there is a neighborhood U of t such that $S(s)$ is true for all $s \in U \cap (t, \infty) \cap \mathbb{T}$.*
- *If $t \in (t_0, \infty) \cap \mathbb{T}$ is left-dense and $S(s)$ is true for all $s \in [t_0, t) \cap \mathbb{T}$, then $S(t)$ is true.*

Then $S(t)$ is true for all $t \in [t_0, \infty) \cap \mathbb{T}$.

Definition 2.2. Δ-derivative of a function $f : \mathbb{T} \to \mathbb{X}$ in a point t, where $\mathbb{X}$ is a linear normed space, is the point $f^{\Delta}(t) \in \mathbb{X}$ (if it exists) such that:

$$\forall_{\varepsilon > 0}\ \exists_{\delta > 0}\ \forall_{s \in B(t,\delta) \cap \mathbb{T}}\ \ \|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)\| \leq \varepsilon |\sigma(t) - s|$$

Proposition 2.3. *If a function f is continuous in t and:*

- *$t = \sigma(t)$, then: $f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}$*
- *$t \neq \sigma(t)$, then: $f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}$*
- *in general, for $t \in \mathbb{T}^{\kappa}$ we have*

$$f^{\Delta}(t) = \lim_{s \to t\ \mathbb{T}} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}$$

where $\mathbb{T}^{\kappa}$ is the set $\mathbb{T}$ without the point $\max \mathbb{T}$ if this point exists and is isolated.

2.2 Δ-differential equations

Definition 2.4. By a local solution of a system of equations:

$$\begin{cases} x^{\Delta}(t) = f(t, x(t)) \\ x(t_0) = x_0 \end{cases} \tag{1}$$

we will mean a continuous function $x\colon \mathbb{T} \cap (a,b) \to \mathbb{X}$ such that $a < \rho(t_0)$, $b > \sigma(t_0)$, $x(t_0) = x_0$ and for all $t \in \mathbb{T}^{\kappa} \cap (a,b)$ equation $x^{\Delta}(t) = f(t, x(t))$ is fulfilled.

Definition 2.5. A solution x_2 is an extension of a solution x_1, if x_1 and x_2 are local solutions of the same system of equations, $\mathrm{Dom}(x_2) \supsetneq \mathrm{Dom}(x_1)$ and $x_2|_{\mathrm{Dom}(x_1)} = x_1$.

If we cannot extend a local solution, then we call it a global solution.

Proposition 2.6. *If for all $t_0 \in \mathbb{T}^{\kappa}$ and $x_0 \in \mathbb{X}$ there exists a unique local solution of system (1), then for all $t_0 \in \mathbb{T}^{\kappa}$ and $x_0 \in \mathbb{X}$ there exists a unique global solution of the same equation $x : \mathbb{T} \cap (a,b) \to \mathbb{X}$, where $\mu(a) = 0$ or $a = -\infty$, and $b - \rho(b) = 0$ or $b = \infty$.*

In analogy to standard local processes on $\mathbb{R}$ we can define a local Δ-process. We have then a formal definition:

Definition 2.7. A continuous function $\Pi\colon M \to \mathbb{X}$ (where $M \subset \mathbb{X} \times \mathbb{T}^2$) is a local Δ-process if:

P1 $\forall_{x \in \mathbb{X}, t \in \mathbb{T}} \exists_{\alpha < t < \beta}\ (\mu(\alpha) = 0 \vee \alpha = -\infty) \wedge (\beta - \rho(\beta) = 0 \vee \beta = \infty) \wedge$
$\wedge \{s \in \mathbb{T}\ ;\ (x,t,s) \in M\} = (\alpha, \beta) \cap \mathbb{T}$,

P2 $\forall_{x \in \mathbb{X}, t \in \mathbb{T}}\ \Pi(x,t,t) = x$,

P3 $\forall_{(x,t,s),(x,t,r) \in M}\ (\Pi(x,t,s), s, r) \in M \wedge \Pi(\Pi(x,t,s), s, r) = \Pi(x,t,r)$.

Definition 2.8. We say that an equation $x^{\Delta}(t) = f(t, x(t))$ generates a local Δ-process Π, if for all $x_0 \in \mathbb{X}$ and $t_0 \in \mathbb{T}$ a function $\Pi(x_0, t_0, \cdot)$ is a global and unique solution of (1) and Π is a local Δ-process.

Analogously as processes on $\mathbb{R}$, a Δ-process induce homeomorphisms along trajectories:

There is also possibility of understanding a solution as a function that fulfills the equation $x^{\Delta}(t) = f(t, x(t))$ $\mathbb{T}$-almost everywhere (that concept had been introduced in [11]). If we accept this definition the next part of this paper needs only nonsignificant changes.

Proposition 2.9. *If an equation $x^{\Delta}(t) = f(t, x(t))$ generates a local Δ-process Π, and if all solutions of the problem*

$$\begin{cases} x^{\Delta}(t) = f(t, x(t)) \\ x(t_0) \in A \end{cases}$$

exist in time t_1, then $\Pi(\cdot, t_0, t_1)|_A$ is a homeomorphism between A and its image.

Proof. Π is continuous so $\Pi(\cdot, t_0, t_1)$ and $\Pi(\cdot, t_1, t_0)$ are continuous on theirs domains which implies what was to prove. □

We will need the preservation of orientation by Π. Below we show a simple theorem which gives an example of a class of functions implying that property.

Definition 2.10. A function $f : \mathbb{T} \times \mathbb{X} \to \mathbb{X}$ is rd-continuous if it is continuous in all $t \in \mathbb{T}$ such that $\mu(t) = 0$, that is in so-called right dense points (this justifies "rd" in the name).

Proposition 2.11. *Let $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ be rd-continuous and Lipschitz continuous (with Lipschitz constant $L(t)$) with respect to the second variable. If for all $t \in \mathbb{T}$ inequality $L(t)\mu(t) < 1$ is fulfilled, then an equation $x^{\Delta}(t) = f(t, x(t))$ generates a local Δ-process Π and for all $t \in \mathbb{T}$ we have that function $\Pi(\cdot, t, \sigma(t))$ preserves an orientation of $\mathbb{R}^n$.*

Proof. An equation $x^{\Delta}(t) = f(t, x(t))$ has a global and unique solution (see [9, p.322, 324]) with a continuous dependence on the initial conditions, so this equation generates a local Δ-process Π. Moreover:

$$\Pi(x, t, \sigma(t)) - \Pi(0, t, \sigma(t)) = x + \mu(t) f(t, x) - (0 + \mu(t) f(t, 0)) =$$

$$= x + \mu(t)(f(t, x) - f(t, 0))$$

so, by $L(t)\mu(t) < 1$, for $x \neq 0$ we have:

$$\langle \Pi(x, t, \sigma(t)) - \Pi(0, t, \sigma(t)), x \rangle > 0$$

which means that vectors $\Pi(x, t, \sigma(t)) - \Pi(0, t, \sigma(t))$ and x are in the same halfspace, so $\Pi(\cdot, t, \sigma(t))$ preserves an orientation of $\mathbb{R}^n$. □

3 Ważewski method for the whole boundary egress set

There are shown two approaches to basic Ważewski Theorem in this chapter, which means: the case of set Ω, for which all trajectories starting from the boundary immediately leaves that set. In this section we assume, that $\mathbb{X} = \mathbb{R}^n$.

3.1 Approach 1.

If local Δ-process Π generated by Δ-equation is not well defined in (x, t_0, t_1), it means, that solution starting in (x, t_0) reaches to boundary of $\mathbb{R}^n$ - infinity (one point compactification of $\mathbb{R}^n$). This observation leads to convenient notation for points (x, t_0, t_1) outside of the domain of local Δ-process Π: $\Pi(x, t_0, t_1) := \infty$.

We will use a function of positively closest point of change of interval charakter of $\mathbb{T}$.

Definition 3.1. Essential forward jump operator is a function $\mathrm{ess}\sigma : \mathbb{T} \to \mathbb{T} \cup \sup \mathbb{T}$ with formula:

$$\mathrm{ess}\sigma(t) := \inf\{s \in \mathbb{T}\ ;\ s > t \wedge (\mu(s) > 0 \vee \mu(t) > 0)\}.$$

Let $\tilde{\Omega}$ be closed subset of $\mathbb{R} \times \mathbb{R}^n$, such that for each $r \in \mathbb{R}$ the set $\tilde{\Omega}_r := \{x \in \mathbb{R}^n\ ;\ (r, x) \in \tilde{\Omega}\}$ is nonempty and bounded, $\partial(\tilde{\Omega}_r)$ is not a retrakt of $\tilde{\Omega}_r$ and $\{r\} \times \partial(\tilde{\Omega}_r)$ is a retrakt of $\partial(\tilde{\Omega})$. We will use curtailment of $\tilde{\Omega}$ to the time scale $\mathbb{T}$:

$$\Omega := \bigcup_{t \in \mathbb{T}} \{t\} \times \tilde{\Omega}_t.$$

Theorem 3.2. *For above set Ω and equation $x^{\Delta}(t) = f(t, x(t))$, which generates local Δ-process Π, if for all $t \in \mathbb{T}$ and for all $s \in (t, \mathrm{ess}\sigma(t)]$ we have $\Pi(cl(\Omega^c)_t, t, s) \subset (\Omega_s)^c$() then for each $t_0 \in \mathbb{T}$ there exists point $x_0 \in \Omega_{t_0}$, such that the solution starting from (t_0, x_0) remain in Ω for every $t \in \mathbb{T}$ bigger than t_0.*

Similar theorem is proved in [7], but they are focused on simple time scales (with values of grainies function equal 0 or bigger than ϵ), and then using inverse systems and analitic means, they obtain general case.

This condition means that starting from the outside of set Ω there is no trajektory such that enters set Ω up to time of essential forward jump of starting time, which means that the whole boundary of Ω is a set of egress points in a specyfic sens.

Proof. We will prove by induction principle for time scales (Proposition 2.1), that $\Pi(\Omega_t, t, s) \subset \Omega_s$ for $s, t \in \mathbb{T}$ where $s \leqslant t$.

Obviously $\Pi(\Omega_t, t, t) = \Omega_t$.

We have ess$\sigma(t) = \sigma(t)$ for right-scattered points, therefor: $\Pi(\mathrm{cl}(\Omega^c)_t, t, \sigma(t)) \in (\Omega_{\sigma(t)})^c$, so $\Pi(\Omega_{\sigma(t)}, \sigma(t), t) \subset \Omega_t$.

For points t in compact interval in time scale, that are not right boundery of that interval we have that ess$\sigma(t)$ is right boundery of that interval. In particular we have ess$\sigma(t) - t > \varepsilon > 0$, so for $s \in (t, t+\varepsilon)$ we have $\Pi(\Omega_t, t, s) \subset \Omega_s$.

For other right-dense points t we know that ther exists sequence $(t_n) \subset \mathbb{T}$ diminishing to t such that $\mu(t_n) > 0$ for all n, for which we have $\Pi(\Omega_{\sigma(t_n)}, \sigma(t_n), t_n) \subset \Omega_{t_n}$, so by continuity of Π we find $\varepsilon > 0$ such that for $s \in (t, t+\varepsilon) \cap \mathbb{T}$ we have $\Pi(\Omega_t, t, s) \subset \Omega_s$.

For left-dense points we obtain needed property also by continuity.

By induction we have $\Pi(\Omega_t, t, s) \subset \Omega_s$ for $s, t \in \mathbb{T}$ where $s \leqslant t$.

Let us fix $t_0 \in \mathbb{T}$. We can choose sequence $(t_n)_{n=0..\infty} \subset \mathbb{T}$ increasing to $\sup \mathbb{T}$ and define $(\Omega^n)_{n=1..\infty}$ by:

$$\Omega^n := \Pi(\Omega_{t_n}, t_n, t_0).$$

We know, that:

$$\Omega^{n+1} = \Pi(\Pi(\Omega_{t_{n+1}}, t_{n+1}, t_n), t_n, t_0) \subset \Omega^n,$$

therefore $(\Omega^n)_{n=1..\infty}$ is descending family of compact sets. Intersection of descending family of nonemty compact sets is nonempty and we choose x_0 in that intersection. We now know, that trajektory starting in (t_0, x_0) is in Ω up to any time t_n, and $t_n \to \sup \mathbb{T}$, so this is the searched trajektory.

□

3.2 Approach 2.

Let $b_i, c_i : \mathbb{T} \to \mathbb{R}$ (for $i = 1..n$) be Δ-differentiable functions where $b_i < c_i$. We define Ω using that functions:

$$\Omega := \{(t, x) \in \mathbb{T} \times \mathbb{R}^n \ ; \ b_i(t) \leqslant x_i \leqslant c_i(t) \text{ for all } i\}$$

and we will use notoation:

$$\partial_{\mathbb{T}}\Omega := \{(t,x) \in \mathbb{T} \times \mathbb{R}^n \; ; \; b_i(t) \leqslant x_i \leqslant c_i(t) \text{ for all } i \\ \text{wherein at least one inequality is equality}\}.$$

All points $p \in \partial_{\mathbb{T}}\Omega$ can be presented in one of the following ways:

$$p = (t, x_1, ..., x_{i-1}, b_i(t), x_{i+1}, ..., x_n) \in \Omega_b^i$$

or

$$p = (t, x_1, ..., x_{i-1}, c_i(t), x_{i+1}, ..., x_n) \in \Omega_c^i.$$

Theorem 3.3. *Let $b_i, c_i : \mathbb{T} \to \mathbb{R}$ be Δ-differentiable and $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ generates local Δ-process Π. If for all $(t,x) \in \Omega_b^i$ we have $f(t,x) < b_i^{\Delta}(t)$ and for all $(t,x) \in \Omega_c^i$ we have $f(t,x) > c_i^{\Delta}(t)$ then for each $t_0 \in \mathbb{T}$ there exists $x_0 \in \Omega_{t_0}$, such that solution starting in (t_0, x_0) remains in Ω for every $t \in \mathbb{T}$ bigger than t_0.*

Proof. (ad absurdum)

Let us notice, that for $(t,x) \in \Omega_b^i$, where $\mu(t) > 0$, we have $\Pi(x,t,\sigma(t)) = x + \int_t^{\sigma(t)} f(\tau,x)\Delta\tau < x + \int_t^{\sigma(t)} b_i^{\Delta}(\tau)\Delta\tau = b_i(\sigma(t))$, and similarly for $\mu(t) = 0$ we have $\Pi(x,t,t+\epsilon) = x + \int_t^{t+\epsilon} f(\tau,x)\Delta\tau < x + \int_t^{t+\epsilon} b_i^{\Delta}(\tau)\Delta\tau = b_i(t+\epsilon)$, which means that trajectories starting in Ω_b^i immiedietly leaves set Ω. By analogy, trajectories starting in Ω_c^i also immiedietly leaves set Ω.

Let us fix $t_0 \in \mathbb{T}$. We extend linearly Ω to continuous tube on $[t_0, \sup \mathbb{T}] \cap \mathbb{R}$:

$\Omega^* := \Omega \cup \{(t,x) \in ([t_0, \sup \mathbb{T}] \cap \mathbb{R} \setminus \mathbb{T}) \times \mathbb{R}^n \; ;$

$$b_i(t_a) + (b_i(t_b) - b_i(t_a))\frac{t - t_a}{t_b - t_a} \leqslant x_i \leqslant c_i(t_a) + (c_i(t_b) - c_i(t_a))\frac{t - t_a}{t_b - t_a} \\ \text{for all } i\}$$

where $t_a, t_b \in \mathbb{T}$ are such that $t_a < t < t_b$ and $(t_a, t_b) \cap \mathbb{T} = \emptyset$.

Naturally, for above t_a and t_b we know that set $\Omega^*_{[t_a,t_b]}$ is convex.

Note that $r : \partial\Omega^* \to \{t_0\} \times \partial\Omega_{t_0}$

$$r(t,x) := (t_0, (b_i(t_0) + \frac{c_i(t_0) - b_i(t_0)}{c_i(t) - b_i(t)}(x_i - b_i(t)))_{i=1..n})$$

This is Theorem from [6], with assumptions given in the the language of Δ-processes.

is a retraction.

Now we will find continuous function from the set Ω_{t_0} to the boundary of Ω^*.

Negation of the thesis means that for all $x \in \Omega_{t_0}$ we have finite time of exit
$t_e(x) := \sup\{t \in \mathbb{T} \ ; \ \forall_{s\in\mathbb{T}\cap[t_0,t]}(t, \Pi(x, t_0, t)) \in \Omega\} < \sup \mathbb{T}$.

If $\mu(t_e(x)) = 0$, then $(t_e(x), \Pi(x, t_0, t_e(x)))$ is in boundary of Ω^*.

If $\mu(t_e(x)) \neq 0$, then $(t_e(x), \Pi(x, t_0, t_e(x))) \in \Omega$ and $(\sigma(t_e(x)), \Pi(, t_0, \sigma(t_e(x)))) \notin \Omega$ and by convexity of $\Omega^*_{[t_e(x),\sigma(t_e(x))]}$ we obtain unique intersection of $\Omega^*_{[t_e(x),\sigma(t_e(x))]}$ with interval connecting points $(t_e(x), \Pi(x, t_0, t_e(x)))$ and $(\sigma(t_e(x)), \Pi(x, t_0, \sigma(t_e(x))))$. Denote this point by $(t^*_e(x), x^*_e)$.

Therefore, we can define function $p\colon \Omega_{t_0} \to \partial\Omega^*$:

$$p(x) := \begin{cases} (t_e(x), \Pi(x, t_0, t_e(x))), & \text{gdy } \mu(t_e(x)) = 0 \\ (t^*_e(x), x^*_e), & \text{gdy } \mu(t_e(x)) > 0. \end{cases}$$

By continuities of Π and tube Ω^* we have continuous dependence $(t^*_e(x), x^*_e)$ and $(t_e(x), \Pi(x, t_0, t_e(x)))$ in respect to x. It is enough to show continuous dependence beetwen $(t^*_e(x), x^*_e)$ and $(t_e(x), \Pi(x, t_0, t_e(x)))$.

For point x_0 such that $t_e(x_0)$ is left-dense and right-scatered and $(t_e(x_0), \Pi(x_0, t_0, t_e(x_0))) \in \Omega$ we have $p(x) \to p(x_0)$ for $x \to x_0$ with the time of exit $t_e(x) < t_e(x_0)$, and $p(x) \to p(x_0)$ for $x \to x_0$ with the time of exit $t_e(x) = t_e(x_0)$. By continuity of Π we can choose small enough neighborhood of x_0, which do not contains another points, so p is continuous in x_0.

By analogy, for left-scattered and rigth-dense points $t_e(x_0)$ we obtain continuity of p in such points. Therefore p is continuous.

Note that function $R\colon \{t_0\} \times \Omega_{t_0} \to \{t_0\} \times \partial\Omega_{t_0}$

$$R(t_0, x) := r(p(x))$$

is a composition of continuous functions, so it is a retraction, which is in contradiction with the construction of set Ω. □

4 Ważewski method for non-whole boundary egress set

In this section there are presented results from [8].

4.1 Notation

Let $B(x,r)$ denote an open ball centered in $x \in \mathbb{R}^2$ and with a radius r, $D(x,r) = \operatorname{cl} B(x,r)$, $S(x,r) = \partial B(x,r)$ and $S^1 := S(0,1)$.

Proposition 4.1 (Shöenflies theorem)**.** *Any homeomorphism $h : S^1 \to h(S^1) \subset \mathbb{R}^2$ can be extended to a homeomorphism $\tilde{h} : \mathbb{R}^2 \to \mathbb{R}^2$.*

In particular, for any homeomorphism $h : S^1 \to h(S^1) \subset \mathbb{R}^2$ there exists a homeomorphism $\hat{h} : D(0,1) \to \hat{h}(D(0,1)) \subset \mathbb{R}^2$ such that the set $\hat{h}(S^1)$ is a boundary of $\hat{h}(D(0,1))$ and the equality $h(x) = \hat{h}(x)$ holds for all $x \in S^1$.

Let $A \subset \mathbb{T} \times \mathbb{R}^2$. Then we define:

$$A_t := \{x \in \mathbb{R}^2 \; ; \; (t,x) \in A\}.$$

Let $\Theta : \mathbb{T} \times S^1 \to \mathbb{R}^2$ be a continuous function such that:

- $\Theta_t : S^1 \to \Theta_t(S^1) \subset \mathbb{R}^2$, where $\Theta_t(x) = \Theta(t,x)$, is a homeomorphism,
- $\Theta(t,s) = \Theta(\sigma(t),s)$.

For all $t \in \mathbb{T}$ let Ω_t be a closure of a bounded open set surrounded by the curve $\Theta(t, S^1)$ and

$$\Omega := \bigcup_{t \in \mathbb{T}} \{t\} \times \Omega_t.$$

For such construction we will say that Ω *is* Θ*-bounded*. In particular Ω can be a constant tube $\Omega = \mathbb{T} \times \Omega_0$, where Ω_0 is homeomorphic to $D(0,1)$.

We consider the following parts of the set Ω:

$$\partial_{\mathbb{T}}\Omega := \Theta(\mathbb{T} \times S^1) = \bigcup_{t \in \mathbb{T}} \{t\} \times \partial(\Omega_t)$$

$$\partial_{\mathbb{T}}\Omega^+ := \bigcup_{t \in \mathbb{T}} \{t\} \times \operatorname{cl}\{x \in \mathbb{R}^2 \; ; \; (t,x) \in \partial_{\mathbb{T}}\Omega \wedge \exists_{r>0} \forall_{y \in B(x,r) \cap \Omega_t} \forall_{\lambda \in (0,1)} \; \lambda x + (1-\lambda)y \in \operatorname{int} \Omega\}$$

It is a well kown property of planes, which is presented for example in [12, pp. 68,72].

In other words, $(\partial_{\mathbb{T}}\Omega^{+})_t$ is a closure of the set of points in $\partial(\Omega_t)$ that have strictly convex neighborhoods in Ω_t.

For any maps $f:\mathbb{T}\times\mathbb{R}^2\to\mathbb{R}^2$ and $g:\mathbb{R}^2\to\mathbb{R}^2$ we define where it makes sense:

$f_t(x) := f(t,x)$,

$\Phi_g(\cdot,\cdot)$, a local flow generated by the equation $y' = g(y)$

$w_g(x)$, a duration of a solution in a local flow Φ_g started in x

We focus our attention at the following subsets of Ω:

- Set of *egress points*:

$$
\begin{aligned}
E := \{&(t,x)\in\partial_{\mathbb{T}}\Omega\ ;\ y' = f_t(y) \text{ generates a local flow } \Phi_{f_t} \\
&\text{and } \Phi_{f_t}(x,(0,s])\not\subset\Omega_t \text{ for any } s\in(0,w_{f_t}(x))\}
\end{aligned}
$$

- Set of *escape points*:

$$
Es := \{(t,x)\in\Omega\ ;\ \mu(t)\neq 0 \text{ and } x+\mu(t)f(t,x)\notin \mathrm{int}\Omega_{\sigma(t)}\}
$$

4.2 Theorems

Now we will prove the main theorem of the paper.

Theorem 4.2. *Let $f:\mathbb{T}\times\mathbb{R}^2\to\mathbb{R}^2$ be a map such that:*

(H0) *equation $x^{\Delta}(t) = f(t,x(t))$ generates a local Δ-process Π,*

(H1) *for all $t\in\mathbb{T}$ a function $\Pi(\cdot,t,\sigma(t))$ preserves an orientation of $\mathbb{R}^2$,*

(H2) *Ω is Θ-bounded (see section 4.1),*

(H3) *there exists a closed set $W\subsetneq S^1$ such that W is not a retract of $D(0,1)$ and $\Theta(\mathbb{T}\times W) = E$,*

(H4) *if $\mu(t)\neq 0$, then*

(H4a) $\Pi(E_t,t,\sigma(t))\cap\Omega_{\sigma(t)} = \emptyset$,

(H4b) $\Pi(int\Omega_t,t,\sigma(t))\cap\partial_{\mathbb{T}}\Omega_{\sigma(t)}\subset E_{\sigma(t)}$.

Then, for all $t_0\in\mathbb{T}$, there exists a point $x_0\in\Omega_{t_0}$ such that the solution starting in (t_0,x_0) remains in Ω for all $t\in\mathbb{T}$ bigger than t_0.

Proof. (ad absurdum)

Let us fix for a while point $t \in \mathbb{T}$ such that $\mu(t) \neq 0$.

By assumption (H4a), we know that $E_t \subset Es_t$.

By definition of Es we also know that $\Pi(E_{\sigma(t)}, \sigma(t), t) \cap \Omega_t \subset \partial(Es_t)$.

By assumptions (H2) and (H3) we know that $\Omega_t = \Omega_{\sigma(t)}$ and $E_t = E_{\sigma(t)}$. This allows us to define the following continuous tube:

$$\tilde{E} = \{(s, x) \in \mathbb{R} \times \mathbb{R}^2 \; ; \; (sup\{t \in \mathbb{T} \; ; \; t < s\}, x) \in E\}.$$

In other words we fill the interstices caused by a time scale.

We will construct a continuous function $w_t \colon Es_t \to [t, \sigma(t)] \times E_t \subset \tilde{E}$ such that:

w1 $\forall_{x \in E_t} \; w_t(x) = (t, x)$

w2 $\forall_{x \in \Pi(E_{\sigma(t)}, \sigma(t), t) \cap \Omega_t} \; w_t(x) = (\sigma(t), \Pi(x, t, \sigma(t)))$

which we will use to construct a continuous function from Ω_{t_0} to $\tilde{E}$ and by that, a retraction from $D(0, 1)$ to W.

Let I be the set of indices of connected components E_t^i of $\Pi(E_t, t, \sigma(t))$ By assumption (H4a) we know that each set E_t^i is contained in a corresponding connected component γ_i of $\Pi(\partial_{\mathbb{T}}\Omega_t, t, \sigma(t)) \setminus \partial_{\mathbb{T}}\Omega_t$. The curve γ_i cuts from $\mathrm{cl}(\Omega_t^c)$ a closed bounded connected set denoted by Es_t^i.

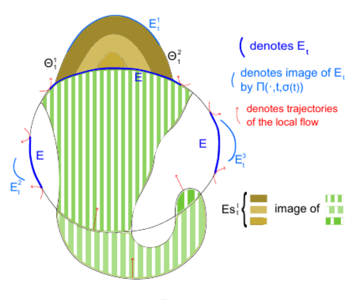

Figure 1

By assumption (H3) we know that $E_t \neq \partial_{\mathbb{T}}\Omega_t$ so a boundary of Es_t^i is a closed curve and a sum of four curves: θ_i^1, E_t^i, θ_i^2 and $\partial_{\mathbb{T}}\Omega \cap Es_t^i$ (each of them homeomorphic to line segments), so it is homeomorphic to S^1 (see Figure 1). By assumption (H1) the sets $\partial_{\mathbb{T}}\Omega_t$ and $\Pi(\partial_{\mathbb{T}}\Omega_t, t, \sigma(t))$

have the same orientation, and therefore E_t^i and $\partial_{\mathbb{T}}\Omega \cap Es_t^i$ have an opposite orientation on the boundary of Es_t^i, so we can parameterize that boundary to obtain a homeomorphism $h_i : \partial T_i \to \partial(Es_t^i)$ such that:

- T_i is a trapezoid with vertices $(0, 0)$, $(1, 0)$, $(a_i, 1)$, $(b_i, 1)$ where $[a_i, b_i] \subset [0, 1]$, ,
- $h_i([0, 1], 0) = E_t^i$,
- $h_i([a_i, b_i], 1) = \partial_{\mathbb{T}}\Omega_t \cap Es_t^i$,
- $\forall_{x\in[a_i,b_i]} h_i(x, 0) = \Pi(h_i(x, 1), t, \sigma(t))$.

By the Shöenflies theorem we can extend h_i to a homeomorphism $\hat{h_i} : T_i \to Es_t^i$. If $\Pi(Es_t, t, \sigma(t)) \setminus \bigcup_{i\in I} Es_t^i \neq \emptyset$, then with other con-

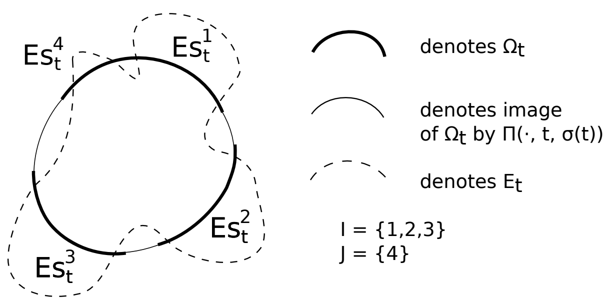

Figure 2

nected components (indexed with elements of some set J) we make similar sets Es_t^j, each of them bounded by a part of $\partial(\Omega_t)$ and a part of $\Pi(\partial(\Omega_t), t, \sigma(t))$, so bounded by a curve homeomorphic to S^1 (see Figure 2). Then we have a homeomorphism $h_j : \partial T_j \to \partial(Es_t^j)$ such that:

- T_j is a triangle with vertices $(1/2, 1/2)$, $(0, 1)$, $(1, 1)$,
- $h_j([0, 1], 1) = \partial_{\mathbb{T}}\Omega_t \cap Es_t^j$.

and again by the Shöenflies theorem we can extend h_j to a homeomorphism $\hat{h_j} : T_j \to Es_t^j$.

With that construction θ_i^1 and θ_i^2 are the images of the side edges of T_i.
In that construction point $h_j(1/2, 1/2)$ is free to choose.

Now we know that $\Pi(Es_t, t, \sigma(t)) \subset \bigcup_{i\in I} Es_t^i \cup \bigcup_{j\in J} Es_t^j$ which is the sum of disjoint sets, therefore we can define $w_t : Es_t \to [t, \sigma(t)] \times E_t \subset \tilde{E}$,

$$w_t(x) := \begin{cases} (t+\mu(t)p_2(\cdot), \Pi(\hat{h_i}(p_1(\cdot),0),\sigma(t),t))(\hat{h_i}^{-1}(\Pi(x,t,\sigma(t)))), \\ \hfill \Pi(x,t,\sigma(t)) \in Es_t^i \ \ (i \in I) \\ (t+\mu(t)p_2(\cdot), \Pi(\hat{h_i}(p_1(\cdot),1),\sigma(t),t))(\hat{h_i}^{-1}(\Pi(x,t,\sigma(t)))), \\ \hfill \Pi(x,t,\sigma(t)) \in Es_t^j \ \ (j \in J) \\ (\sigma(t), \Pi(x,t,\sigma(t))) \\ \hfill \Pi(x,t,\sigma(t)) \in E_{\sigma(t)} \end{cases}$$

where p_1 and p_2 are projections respectively onto the first and second variables. By construction it is a continuous function.

Notice that for $x \in E_t$ there exists a unique $i \in I$ such that $\Pi(x,t,\sigma(t)) \in E_t^i$, and consequently $y := \hat{h}_i^{-1}(\Pi(x,t,\sigma(t))) \in [0,1] \times \{0\}$. Since $p_1(y) = y$ and $p_2(y) = 0$, we obtain that $w_t(x) = (t, \Pi(\hat{h}_i(y), \sigma(t), t)) = (t, x)$. Hence property **w1** is satisfied.

Moreover, if $x \in \Pi(E_{\sigma(t)}, \sigma(t), t) \cap \Omega_t$, then $\Pi(x,t,\sigma(t)) \in E_{\sigma(t)}$, so also property **w2** is fulfilled.

Falsity of thesis means that for every $x \in \Omega_{t_0}$ we have:

$t_e(x) := \sup\{t \in \mathbb{T} \,;\, \forall_{s\in\mathbb{T}\cup[t_0,t]} (t, \Pi(x,t_0,t)) \in \Omega\} < \sup \mathbb{T}$.

If $\mu(t_e(x)) = 0$, then $(t_e(x), \Pi(x,t_0,t_e(x)))$ is already in $E \subset \tilde{E}$.

If $\mu(t_e(x)) \neq 0$, then $(t_e(x), \Pi(x,t_0,t_e(x))) \in \Omega$ and $(\sigma(t_e(x)), \Pi(x,t_0,\sigma(t_e(x)))) \notin \Omega$. So we can use the function $w_{t_e(x)}$ to it.

Therefore, we can define $r : \Omega_{t_0} \to \tilde{E}$,

$$r(x) := \begin{cases} (t_e(x), \Pi(x,t_0,t_e(x))), & \text{if } \mu(t_e(x)) = 0, \\ w_{t_e(x)}(\Pi(x,t_0,t_e(x))), & \text{if } \mu(t_e(x)) \neq 0. \end{cases}$$

Take any point x such that $\mu(t_e(x)) = 0$.

Then for each $\varepsilon > 0$ there exists $\tau \in \mathbb{T}$ such that $0 < \tau - t_0 < \varepsilon$ and $\Pi(x,t_0,\tau) \notin \Omega_\tau$ so, by the continuity of Π, for each $\varepsilon > 0$ there exist $\delta > 0$ and $\tau \in \mathbb{T}$ such that $0 < \tau - t_0 < \varepsilon$ and $\Pi(B(x,\delta), t_0, \tau) \cap \Omega_\tau = \emptyset$.

Therefore

$$\forall_{\varepsilon>0} \exists_{\delta>0} \forall_{y\in B(x,\delta)} \ t_e(y) < t_e(x) + \varepsilon.$$

Similarly we show that for each point x such that $t_e(x) - \rho(t_e(x)) = 0$ we have

$\forall_{\varepsilon>0} \exists_{\delta>0} \forall_{y\in B(x,\delta)} \ t_e(y) > t_e(x) - \varepsilon.$

Using a continuity of Π we get a continuity of r in all x such that $t_e(x)$ is a dense point of $\mathbb{T}$. Furthermore, properties **w1** and **w2** guarantee a continuity of r in every point x such that $\mu(t_e(x)) \neq 0$ or $t_e(x)-\rho(t_e(x)) \neq 0$.

Hence r is continuous for all points in Ω_{t_0}.

By the Shöenflies theorem we can extend Θ_t to $\hat{\Theta}_t : D(0,1) \to \Omega_t$ for every $t \in \mathbb{T}$. Using this we define a map $R : D(0,1) \to W$:

$$R(y) := \Theta^{-1}_{t_e(\hat{\Theta}_{t_0}(y))} \circ p_2 \circ r(\hat{\Theta}_{t_0}(y)).$$

For all $y \in W$ we have $R(y) = \Theta^{-1}_{t_0} p_2 r(\Theta_{t_0}(y)) = \Theta^{-1}_{t_0}(\Pi(\Theta_{t_0}(y), t_0, t_0)) = y$ and, by the continuity of r, we get that R is a retraction, what contradicts assumption (H3).

□

The geometric assumption (H4') in the next theorem corresponds to the assumption (H4) and may occure to be easier to check.

Theorem 4.3. *Assume that:*

(H0) *equation $x^{\Delta}(t) = f(t, x(t))$ generates a local Δ-process Π,*

(H1) *for all $t \in \mathbb{T}$ a function $\Pi(\cdot, t, \sigma(t))$ preserves an orientation of $\mathbb{R}^2$,*

(H2) *Ω is Θ-bounded,*

(H3') *there exists a closed set $W \subsetneq S^1$ such that W is not a retract of $D(0,1)$ and $\Theta(\mathbb{T} \times W) = \partial_{\mathbb{T}}\Omega^+ = E$,*

(H4') *if $\mu(t) \neq 0$, then $\Omega_t \subset x + T_{\Omega_t}(x)$ for all $x \in int_{\partial_{\mathbb{T}}\Omega_t} E_t$, where $T_{\Omega_t}(x)$ is a Bouligand tangent cone of the set $\Omega_t \in \mathbb{R}^2$ in a point x ($T_K(x) = \{v \in \mathbb{X}\, ;\, \liminf_{h \to 0^+} \frac{d(x+hv,K)}{h} = 0\}$).*

Then, for all $t_0 \in \mathbb{T}$, there exists a point $x_0 \in \Omega_{t_0}$ such that a solution starting in (t_0, x_0) remains in Ω for all $t \in \mathbb{T}$ bigger than t_0.

Proof. To use Theorem 4.2 it is sufficient to show that, if $\mu(t) \neq 0$, then $\Pi(E_t, t, \sigma(t)) \cap \Omega_{\sigma(t)} = \emptyset$ and $\Pi(\mathrm{int}\Omega_t, t, \sigma(t)) \cap \partial_{\mathbb{T}}\Omega_{\sigma(t)} \subset E_{\sigma(t)}$.

Let us fix $t \in \mathbb{T}$ such that $\mu(t) \neq 0$.

We have that $\Pi(x, t, \sigma(t)) = x + \mu(t) f_t(x)$ and, for all $x \in E_t$, vectors $f_t(x)$ are directed outside the set Ω_t so a local strict convexity of points in E_t (assumption (H3')) guarantees that each connected component $E_{t,i}$

of E_t has no common points with $\Pi(E_{t,i}, t, \sigma(t))$. Moreover, assumption (H4') ensures that the set $\Pi(E_{t,i}, t, \sigma(t))$ is outside of the rest of $\Omega_{\sigma(t)}$, so the first part is fulfilled.

A connected component $I_{t,i}$ of $\partial_{\mathbb{T}}\Omega_t \setminus E_t$ has only points without strictly convex neighborhoods in Ω_t. All of that points are not egress points in a local flow, so $f_t(x)$ are directed inside the set Ω_t. If there were $y \in I_{t,i} \cap \Pi(I_{t,i}, \sigma(t), t)$ and $[y, \Pi(y, t, \sigma(t))] \not\subset \partial_{\mathbb{T}}\Omega_t$, then $I_{t,i}$ would be a part of a spiral shaped curve which end is a beginning of a part of E_t, which would contradict assumption (H4'). Therefore there exists a small enough neighborhood $O_{t,i}$ of $I_{t,i}$ such that an image of $O_{t,i} \cap \mathrm{int}\Omega_t$ has no common points with $I_{t,i}$.

$\partial_{\mathbb{T}}\Omega_t$ is homeomorphic to $\Pi(\partial_{\mathbb{T}}\Omega_t, t, \sigma(t))$ so, if $E_{t,i}$ borders on $I_{t,j}$, then their images have to border as well. Therefore an image of $I_{t,j}$ cuts out subset Ω_t^j of Ω_t that contains $I_{t,j}$. Moreover the image of $\partial_{\mathbb{T}}\Omega_t$ does not have selfintersections so, in particular, an image of $I_{t,j}$ is the only part of the image of Ω_t that can have common points with Ω_t^j. It means that

$$\Pi(\mathrm{int}\Omega_t, t, \sigma(t)) \cap (\partial_{\mathbb{T}}\Omega_{\sigma(t)} \setminus E_{\sigma(t)}) = \emptyset$$

what was needed to prove.

□

Example 4.4. Let $\mathbb{T} = \bigcup_{n\in\mathbb{N}} [2n, 2n+1]$ and $f(t,(x,y)) = \left(\frac{e^{-t}(1-|y|\sin(t\pi))-2x}{3}, \frac{2y+\sin x}{5}\right)$. We are interested in existence of trajectory convergent to $(0,0)$.

We will want to use Theorem 4.2 taking $\Omega := \bigcup_{n\in\mathbb{N}} \bigcup_{t\in[2n,2n+1]} \{t\} \times \{(x_1, x_2) \in \mathbb{R}^2 \,;\, -e^{(n-t)/3} \leqslant x_i \leqslant e^{(n-t)/3}\}$.

Firstly, for $t = 2n+1$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$ we have

$$\|f(t,(x_1,y_1)) - f(t,(x_2,y_2))\| =$$

$$= \|(2(x_2-x_1)/3, (2(y_1-y_2)+\sin(x_1)-\sin(x_2))/5)\| \leq$$

$$\leq \|(2/3, 3/5)\| \, \|(x_1-x_2, y_1-y_2)\| < \|(x_1,y_1)-(x_2,y_2)\|,$$

therefor we have $L(t)\mu(t) < 1$, so assumptions of Proposition 2.11 are met.

The set Ω is selected so that $\Omega_t = \Omega_{\sigma(t)}$ and is Θ-bounded, where $\Theta(t,(x,y)) = \frac{e^{(n-t)/3}}{\sup\{|x|,|y|\}}(x,y)$, for $t \in [2n, 2n+1]$.

We will find E.

For all $t \in [2n, 2n+1]$ and $-e^{(n-t)/3} \leq x \leq e^{(n-t)/3} = |y|$ we have $(t,(x,y)) \in \partial_{\mathbb{T}}\Omega$ and $|\sin(x)| < |y|$, therefore these are the egress points. Whereas for each $t \in [2n, 2n+1]$ i $-e^{(n-t)/3} < y < e^{(n-t)/3} = |x|$ we have $(t,(x,y)) \in \partial_{\mathbb{T}}\Omega$ and $|e^{-t}(1-|y|\sin(t\pi))| \leqslant e^{-t} < e^{(n-t)/3} = |x|$, therefore vectors $f(t,(x,y))$ areare directed to the center of set Ω_t and $\left|\frac{e^{-t}(1-|y|\sin(t\pi))-2x}{3}\right| > |\frac{1}{3}x| = |\frac{d}{dt}e^{(n-t)/3}|$, thus these points are entry points.

For $t = 2n+1$ i $(x,y) \in E_t$ we have $\Pi((x,y),t,\sigma(t)) = (x+\frac{e^{-t}-2x}{3}, y+\frac{2y+\sin(x)}{5}) \notin \Omega_{\sigma(t)}$, so the assumption (H4a) is met.

For $t = 2n+1$ i $(x,y) \in \Omega_t$ we have similarly: $\Pi((x,y),t,\sigma(t)) = (x+\frac{e^{-t/3}-2x}{3}, y+\frac{2y+\sin(x)}{5})$, therefore the first coordinate is inside the segment $[\frac{e^{-t/3}-e^{n/3-t/3}}{3}, \frac{e^{-t/3}+e^{n/3-t/3}}{3}]$, which means that there are no common points with $\partial_{\mathbb{T}}\Omega_t \setminus E_t$, so assumption (H4b) is met too.

All assumptions are satisfied, therefore there exists trajectory remaining in the set Ω, which is convergent to $(0,0)$ (from the selection of the set Ω).

4.3 Remarks

At first we notice that holes homeomorphic to balls in Ω_t are available in Theorem 4.2. Indeed, for n holes we can consider: $\Theta : \mathbb{T} \times \bigoplus_{j=0}^{n} S^1 \to \mathbb{R}^2$ such that $\Theta(\{t\} \times \bigoplus_{k=0}^{n} S^1)$ is homeomorphic to $S(0,1) \cup \bigcup_{k=1}^{n} S((0,(k-1)/n), 1/3n))$.

In the second remark we observe that properties of a local Δ-process Π are essential, not of f itself, so in all approaches we can change our understanding of a solution of $x^{\Delta}(t) = f(t,x(t))$ and treat it as a function that fulfills the equation $\mathbb{T}$-almost everywhere (in a Sobolev space on a time scale). Moreover, we can change assumptions to a $\mathbb{T}$-almost everywhere form. It is important when we look for possible generalizations to differential inclusions or multivalued Δ-processes.

A proof technique presented in Section 4 cannot be repeated in higher dimensions because the Shöenflies theorem does not raise up to them. The following open problem appears:

That concept had been introduced in [11].

Open problem:

Is it possible to use in higher dimensional spaces the geometric idea presented in the proof of Theorem 4.2 under some additional restrictions to Θ or Π?

Nevertheless, this geometric idea opens new perspectives in the Ważewski retract method on time scales and allows us to study more classes of systems (for example hyperbolic systems).

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Weak local Nash equilibrium - part II

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In the paper “Weak local Nash equilibrium” we define a concept of local equilibrium to non-cooperative games and we prove its existence applying the Lefschetz fixed point theorem. We was inspired by the original Nash's theorem and his proof.

1 Introduction

In the paper “Weak local Nash equilibrium” we define a concept of local equilibrium to non-cooperative games and we prove its existence applying the Lefschetz fixed point theorem. We was inspired by the original Nash's theorem and his proof.

The concept of Nash equilibrium says that an equilibrium for payoff functions

$$p_1, p_2, \dots, p_n : S = S_1 \times S_2 \times \cdots \times S_n \to \mathbb{R}$$

is a point $\tilde{s} = (\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_n) \in S$ such that, for each $i \in \{1, 2, \dots, n\}$,

$$p_i(\tilde{s}_1, \dots, \tilde{s}_{i-1}, s_i, \tilde{s}_{i+1}, \dots, \tilde{s}_n) \leq p_i(\tilde{s}), \text{ for all } s_i \in S_i.$$

Nash proved that:

Theorem 1.1 (Nash's Theorem). *Let $S_1, \dots, S_n$ be compact convex subsets of an Euclidean space. Suppose that $p_1, \dots, p_n : S = S_1 \times \cdots \times S_n \to \mathbb{R}$ are maps such that, for each $i = 1, \dots, n$, $p_i(s_1, \dots, s_n)$ is linear (afim) as a function of s_i. Then there exists at least one equilibrium to $p_1, \dots, p_n$.*

The proof is the following: let $S_i \subset \mathbb{R}^{d_i}$, where d_i is the dimension of S_i. Thus, $S \subset \mathbb{R}^d$, where $d = d_1 + \cdots + d_n$. From the hypothesis, the payoff functions are of the type

$$p_i(s) = v_i(s) \cdot s_i + u_i(s)$$

where $v_i : S \to \mathbb{R}^{d_i}$ and $u_i : S \to \mathbb{R}$ are maps which don't depend on the coordinate s_i, $i = 1, \ldots, n$. Let $v : S \to \mathbb{R}^d$ be the vector field defined by $v(s) = (v_1(s), \ldots, v_n(s))$. Let $r : \mathbb{R}^n \to S$ be the natural retraction that assigns each point $p \in \mathbb{R}^n$ to the point $r(p) \in S$ which realizes the distance of p to S. Finally, let $f : S \to S$ be defined by $f(s) = r(s+v(s))$. Then, one can shown that $\tilde{s} \in S$ is a Nash equilibrium to $p_1, \ldots, p_n$ if and only if $\tilde{s}$ is a fixed point of f. Note that the existence of a fixed point to f is assured by Brouwer's fixed point theorem.

Based on the above proof, we investigated the existence of equilibrium in the context that the spaces of strategies are compact ENR's, not necessarily convex. This means that each space S_i is a subset of some euclidean space $\mathbb{R}^{d_i}$ and there is an open neighborhood V_i of S_i in $\mathbb{R}^{d_i}$ and a retraction $r_i : V_i \to S_i$. From this research, the following definitions arise.

Definition 1. *Let $(S_1, d_1), \ldots, (S_n, d_n)$ be metric spaces and $p_1, \ldots, p_n : S_1 \times \cdots \times S_n \to \mathbb{R}$ real functions. We say that $\tilde{s} = (\tilde{s}_1, \ldots, \tilde{s}_n) \in S$ is a* **weak local equilibrium (abbrev., w.l.e.)** *for $p_1, \ldots, p_n$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$p_i(\tilde{s}_1, \ldots, \tilde{s}_{i-1}, s_i, \tilde{s}_{i+1}, \ldots, \tilde{s}_n) \leq p_i(\tilde{s}) + \varepsilon d_i(s_i, \tilde{s}_i),$$

for every $s_i \in B(\tilde{s}_i, \delta)$, $i = 1, 2, \ldots, n$, where $B(\tilde{s}_i, \delta)$ denotes the open ball with center in $\tilde{s}_i$ and radius $\delta > 0$ in (S_i, d_i).

Definition 2. *We say that a subset X of $\mathbb{R}^m$ has the* **property of convenient retraction (abbrev., p.c.r.)** *if there exists a retraction $r : V \to X$, where V is an open neighborhood of X in $\mathbb{R}^m$, satisfying: given $x_0 \in V$ and $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\langle x_0 - r(x_0), x - r(x_0) \rangle \leq \varepsilon \| x - r(x_0) \|,$$

for all $x \in X$ with $\|x - r(x_0)\| < \delta$, where $\langle\ ,\ \rangle$ is the usual inner product in $\mathbb{R}^m$ and $\| \cdot \|$ is the norm induced by it. In this case, we say that $r : V \to X$ is a convenient retraction.

Example 1. *Every closed convex subset K of $\mathbb{R}^m$ has the p.c.r.. In fact, there is a natural retraction $r : \mathbb{R}^m \to K$ such that to each $x \in \mathbb{R}^m$ assigns the point $r(x) \in K$ which realizes the distance of x to K. This retraction satisfies $\langle x_0 - r(x_0), x - r(x_0)\rangle \leq 0$ for every $x_0 \in \mathbb{R}^m$ and $x \in K$.*

Example 2 ([3], Proposition 4.3)**.** *Every submanifold M of $\mathbb{R}^n$, of class C^2, with or without boundary, has the p.c.r..*

Let X be a closed subset of the Euclidean space $\mathbb{R}^n$ and let V be an open neighborhood of X in $\mathbb{R}^n$. A map $r : V \to X$ is called a proximative retraction (or metric projection) if

$$\|r(y) - y\| = \mathrm{dist}(y, X), \text{ for every } y \in V,$$

where

$$\mathrm{dist}(y, X) = \inf\{\|x - y\| \mid x \in X\}$$

is the distance of y to X.

Evidently, every proximative retraction is a retraction map but not conversely.

A compact subset $K \subset \mathbb{R}^n$ is called a proximative neighborhood retract (written $K \in \mathrm{PANR}$) if there exists an open neighborhood V of K in $\mathbb{R}^n$ and a proximative retraction $r : V \to K$.

We have the following statement:

Example 3 ([2])**.** *Let K be a compact subset of $\mathbb{R}^n$. If $K \in$ PANR then K is an* ENR *with the p.c.r..*

In the previous paper, we was able to prove the following result.

Theorem 1.2 ([2])**.** *Let $p_1, \ldots, p_n : S_1 \times \ldots \times S_n \to \mathbb{R}$ be maps, where each $S_i \subset \mathbb{R}^{m_i}$ is a compact ENR with the p.c.r.. Also, suppose $p_i(s_1, \ldots, s_n)$ continuously differentiable in a neighborhood of s_i when the other variables are kept fixed, $i = 1, 2, \ldots, n$. If $\chi(S_i) \neq 0$ for $i = 1, 2, \ldots, n$ then $p_1, p_2, \ldots, p_n$ have at least one w.l.e..*

Our goal in this paper is to prove a more general version of Theorem 1.2 changing the hypothesis of the continuously differentiable on the payoffs by a weaker hypothesis.

2 Preliminaires

In this section, we define a concept of an upper semi differentiable(u.s.d.) function.

The open ball in $\mathbb{R}^n$ with center in x_0 and radius $r > 0$ will be denoted by $B(x_0, r)$.

Definition 3. *Let $f : A \to \mathbb{R}$ be a function, where A is an open nonempty subset of $\mathbb{R}^n$. Given $x_0 \in A$, we say that f is upper semi differrentiable(u.s.d.) at x_0 if there exists at least one point $v \in \mathbb{R}^n$ together with a function $r : B(0, \varepsilon) \to \mathbb{R}$ such that $\lim_{h \to 0} \frac{r(h)}{\|h\|} = 0$ and*

$$f(x_0 + h) \leq f(x_0) + v \cdot h + r(h)$$

for every h such that $x_0 + h \in A$.

We denote by $DSf(x_0)$ the set of such vectors v.

Example 4. *If $f : A \to \mathbb{R}$ is differentiable at x_0 then f is u.s.d.. Moreover, $DSf(x_0) = \{f'(x_0)\}$. In fact, suppose $v \in \mathbb{R}^n$ and $r : B(0, \varepsilon) \to \mathbb{R}$ such that $\lim_{h \to 0} \frac{r(h)}{h} = 0$ and $f(x_0 + h) \leq f(x_0) + v \cdot h + r(h)$ for every h. Thus, for $0 < t < \varepsilon$,*

$$\frac{f(x_0 + te_i) - f(x_0)}{t} \leq v \cdot e_i + \frac{r(te_i)}{t}.$$

It follows that

$$\frac{\partial f}{\partial x_i}(x_0) = \lim_{t \to 0^+} \frac{f(x_0 + te_i) - f(x_0)}{t} \leq v \cdot e_i.$$

On the other hand, for $-\varepsilon < t < 0$,

$$\frac{f(x_0 + te_i) - f(x_0)}{t} \geq v \cdot e_i + \frac{r(te_i)}{t}.$$

It follows that

$$\frac{\partial f}{\partial x_i}(x_0) = \lim_{t \to 0^-} \frac{f(x_0 + te_i) - f(x_0)}{t} \geq v \cdot e_i.$$

Therefore, $\frac{\partial f}{\partial x_i}(x_0) = v \cdot e_i$.

Thus, $v = f'(x_0) = \left(\frac{\partial f}{\partial x_1}(x_0), \ldots, \frac{\partial f}{\partial x_n}(x_0) \right)$.

The next result shows that the set $DSf(x_0)$ is convex.

Theorem 2.1. *If f is u.s.d. at x_0 then $DSf(x_0)$ is a convex subset of $\mathbb{R}^n$.*

Proof. Let $v_1, v_2 \in DSf(x_0)$ be arbitraires and let $r_1, r_2 : B(0, \varepsilon) \to \mathbb{R}$ be such that

$$\begin{array}{rcl} f(x_0+h) & \leq & f(x_0) + v_1 \cdot h + r_1(h) \\ f(x_0+h) & \leq & f(x_0) + v_2 \cdot h + r_2(h) \end{array}$$

with $\lim\limits_{h\to 0} \dfrac{r_1(h)}{\|h\|} = \lim\limits_{h\to 0} \dfrac{r_2(h)}{\|h\|} = 0$.

Let $v = \alpha v_1 + (1-\alpha)v_2$, with $\alpha \in (0,1)$. We have

$$\begin{array}{rcl} f(x_0+h) & = & \alpha f(x_0+h) + (1-\alpha) f(x_0+h) \\ & \leq & \alpha f(x_0) + \alpha v_1 \cdot h + \alpha r_1(h) + (1-\alpha) f(x_0) \\ & & +(1-\alpha) v_2 \cdot h + (1-\alpha) r_2(h) \\ & = & f(x_0) + v \cdot h + \alpha r_1(h) + (1-\alpha) r_2(h). \end{array}$$

Since

$$\lim_{h\to 0} \frac{\alpha r_1(h) + (1-\alpha) r_2(h)}{\|h\|} = \alpha \lim_{h\to 0} \frac{r_1(h)}{\|h\|} + (1-\alpha) \lim_{h\to 0} \frac{r_2(h)}{\|h\|} = 0,$$

it follows that $v \in DSf(x_0)$.

Therefore, $DSf(x_0)$ is convex. $\square$

In the next theorems, we give conditions to $DSf(x_0)$ be compact.

Theorem 2.2. *Let $f : J \to \mathbb{R}$ be a function, where $J \subset \mathbb{R}$ is open interval, and let $x_0 \in J$. Suppose the existence of the right and left-hand limits*

$$c = \lim_{h\to 0^+} \frac{f(x_0+h) - f(x_0)}{h}$$

and

$$d = \lim_{h\to 0^-} \frac{f(x_0+h) - f(x_0)}{h}$$

Then, f is u.s.d. if and only if $c \leq d$. Moreover, $DSf(x_0) = [c, d]$.

Proof. Suppose f u.s.d. at x_0 and let $v \in DSf(x_0)$. If $0 < h < \varepsilon$, we have

$$\frac{f(x_0+h)-f(x_0)}{h} \leq v + \frac{r(h)}{h},$$

following that $c = \lim\limits_{h\to 0^+} \dfrac{f(x_0+h)-f(x_0)}{h} \leq v$.

Analogously, if $-\varepsilon < h < 0$, we have

$$\frac{f(x_0+h)-f(x_0)}{h} \geq v + \frac{r(h)}{h},$$

following that $d = \lim\limits_{h\to 0^-} \dfrac{f(x_0+h)-f(x_0)}{h} \geq v$.

Therefore, $c \leq d$.

On the other hand, suppose $c \leq d$. Note that, above, we show that $DSf(x_0) \subset [c,d]$. Now, to conclude that $DSf(x_0) = [c,d]$, since $DSf(x_0)$ is convex, it is sufficient to show that $c, d \in DSf(x_0)$.

Define $r(h) = \begin{cases} f(x_0+h)-f(x_0)-ch & \text{se } h>0 \\ 0 & \text{se } h=0 \\ f(x_0+h)-f(x_0)-dh & \text{se } h<0 \end{cases}$

Then $\lim\limits_{h\to 0} \dfrac{r(h)}{h} = 0$. Moreover, for $h > 0$, we have

$$f(x_0+h) = f(x_0) + ch + f(x_0+h) - f(x_0) - ch$$

and, for $h < 0$, we have

$$f(x_0+h) = f(x_0) + ch + f(x_0+h) - f(x_0) - ch \leq f(x_0) + ch + \\ + f(x_0+h) - f(x_0) - dh$$

Therefore, $c \in DSf(x_0)$.

Analogously, for $h > 0$, we have

$$f(x_0+h) = f(x_0) + dh + f(x_0+h) - f(x_0) - dh \leq f(x_0) + dh + \\ + f(x_0+h) - f(x_0) - ch$$

and for $h < 0$,

$$f(x_0 + h) = f(x_0) + dh + f(x_0 + h) - f(x_0) - dh$$

Therefore, $d \in DSf(x_0)$. $\square$

Example 5. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x, & \text{if } x < 0 \\ -x, & \text{if } x \geq 0 \end{cases}$. The function f is u.s.d. at 0. In fact, we have*

$$\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = -1 < 1 = \lim_{h \to 0^-} \frac{f(h) - f(0)}{h}$$

Then, by Theorem 2.2, f is u.s.d. at 0 and $DSf(0) = [-1, 1]$.

Notation: Let $f : A \to \mathbb{R}$ be a map, where A is an open subset of $\mathbb{R}^n$. Let $x_0 \in A$. We denote the right-hand partial derivatives and the left-hand partial derivatives, respectively, by

$$\frac{\partial f^+}{\partial x_i}(x_0) = \lim_{t \to 0^+} \frac{f(x_0 + te_i) - f(x_0)}{t}$$

and

$$\frac{\partial f^-}{\partial x_i}(x_0) = \lim_{t \to 0^-} \frac{f(x_0 + te_i) - f(x_0)}{t}$$

$i = 1, \ldots, n$

Theorem 2.3. *Let $f : A \to \mathbb{R}$ be a map, $A \subset \mathbb{R}^n$ open. Suppose well defined the right-hand and the left-hand partial derivatives of f at every $x_0 \in A$. Also, suppose the functions*

$$\frac{\partial f^+}{\partial x_i}, \frac{\partial f^-}{\partial x_i} : A \to \mathbb{R}$$

continuous and that

$$\frac{\partial f^+}{\partial x_i}(x_0) \leq \frac{\partial f^-}{\partial x_i}(x_0), \ \forall \ x_0 \in A,$$

$i = 1, \ldots, n$. Then, f is u.s.d. and

$$DSf(x_0) = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n],$$

where $a_i = \dfrac{\partial f^+}{\partial x_i}(x_0)$*,* $b_i = \dfrac{\partial f^-}{\partial x_i}(x_0)$*,* $i = 1, \ldots, n$*. Thus,* $DSf : A \multimap \mathbb{R}^n$ *is an u.s.c. multivalued map with convex compact values.*

Proof. Given $x_0 \in A$, let $a_i = \dfrac{\partial f^+}{\partial x_i}(x_0)$, $b_i = \dfrac{\partial f^-}{\partial x_i}(x_0)$, $i = 1, \ldots, n$. The technique used to prove that

$$DSf(x_0) \subset [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$$

is the same used in Theorem 2.2: let $v = (v_1, \ldots, v_n) \in DSf(x_0)$ arbitrary. Thus,

$$f(x_0 + h) \leq f(x_0) + v \cdot h + r(h),$$

with $\lim_{h \to 0} \dfrac{r(h)}{\|h\|} = 0$. In particular, if $h = te_i$ then

$$f(x_0 + te_i) \leq f(x_0) + tv \cdot e_i + r(te_i),$$

with $\lim_{h \to 0} \dfrac{r(te_i)}{t} = 0$. It follows that, for every $t > 0$,

$$\frac{f(x_0 + te_i) - f(x_0)}{t} \leq v_i + \frac{r(te_i)}{t}.$$

Therefore

$$a_i = \frac{\partial f^+}{\partial x_i}(x_0) \leq v_i.$$

Also, for every $t < 0$, we have

$$\frac{f(x_0 + te_i) - f(x_0)}{t} \geq v_i + \frac{r(te_i)}{t}.$$

Therefore,

$$b_i = \frac{\partial f^-}{\partial x_i}(x_0) \geq v_i.$$

Hence, $v \in [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$.

Since $DSf(x_0)$ is convex, in order to prove the equality

$$DSf(x_0) = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$$

it is sufficient to show that each vertex of that parallelepiped is contained in $DSf(x_0)$.

To elucidate, we will write the proof to the case $n = 2$ and for the vertex (a_1, a_2). The general case is analogous.

Let $x_0 = (x_1, x_2)$ and $h = (h_1, h_2)$. We need to show that

$$f(x_1 + h_1, x_2 + h_2) - f(x_1, x_2) - h_1 a_1 - h_2 a_2 \leq r(h)$$

with $\lim\limits_{h \to 0} \dfrac{r(h)}{\|h\|} = 0$.

Consider the functions $g(y) = f(x_1 + h_1, y)$ and $l(x) = f(x, x_2)$. Note that

$$\begin{aligned}
\frac{\partial g^+}{\partial y}(x_2) &= \frac{\partial f^+}{\partial x_2}(x_1 + h_1, x_2) \\
\frac{\partial g^-}{\partial y}(x_2) &= \frac{\partial f^-}{\partial x_2}(x_1 + h_1, x_2) \\
\frac{\partial l^+}{\partial x}(x_1) &= \frac{\partial f^+}{\partial x_1}(x_1, x_2) \\
\frac{\partial l^-}{\partial x}(x_1) &= \frac{\partial f^-}{\partial x_1}(x_1, x_2)
\end{aligned}$$

>From Theorem 2.2, g and l are u.s.d.. Moreover,

$$DSg(x_2) = \left[\frac{\partial f^+}{\partial x_2}(x_1 + h_1, x_2), \frac{\partial f^-}{\partial x_2}(x_1 + h_1, x_2)\right]$$

and

$$DSl(x_1) = \left[\frac{\partial f^+}{\partial x_1}(x_1, x_2), \frac{\partial f^-}{\partial x_1}(x_1, x_2)\right].$$

Thus,

$$\begin{aligned}
g(x_2 + h_2) - g(x_2) - h_2 \frac{\partial f^+}{\partial x_2}(x_1 + h_1, x_2) &\leq r_1(h_2) \\
l(x_1 + h_1) - l(x_1) - h_1 \frac{\partial f^+}{\partial x_1}(x_1, x_2) &\leq r_2(h_1)
\end{aligned}$$

with $\lim\limits_{x \to 0} \dfrac{r_2(x)}{x} = \lim\limits_{y \to 0} \dfrac{r_1(y)}{y} = 0$.

We have that

$$
\begin{aligned}
& f(x_1+h_1,x_2+h_2)-f(x_1,x_2)-h_1a_1-h_2a_2 = \\
& g(x_2+h_2)-g(x_2)-h_2\frac{\partial f^+}{\partial x_2}(x_1+h_1,x_2)+l(x_1+h_1)-l(x_1)- \\
& -h_1\frac{\partial f^+}{\partial x_1}(x_1,x_2)+\ h_2\left[\frac{\partial f^+}{\partial x_2}(x_1+h_1,x_2)-\frac{\partial f^+}{\partial x_2}(x_1,x_2)\right] \\
& \leq\ r(h)
\end{aligned}
$$

where $r(h) = r_1(h_2)+r_2(h_1)+h_2\left[\frac{\partial f^+}{\partial x_2}(x_1+h_1,x_2)-\frac{\partial f^+}{\partial x_2}(x_1,x_2)\right]$.
Now, it is easy to see that $\lim\limits_{h\to 0}\frac{r(h)}{\|h\|}$. □

3 The main theorem

In this section, we will stablish a generalization of the Theorem 1.2. It is the following:

Theorem 3.1. *Let $p_1,\dots,p_n : S_1\times\ldots\times S_n \to \mathbb{R}$ be maps, where each $S_i\subset\mathbb{R}^{m_i}$ is a compact ENR with the p.c.r.. Also, suppose that $p_i(s_1,\dots,s_i,\dots,s_n)$ as a function of $s_i=(s_1^1,\dots,s_1^{m_i})$ satisfies:*

- *The map $x_i \longmapsto p(s_{-i},x_i)$ can be continuously defined on a neighborhood V_i of S_i. The symbol (s_{-i},x_i) denotes the point $(s_1,\dots,s_{i-1},x_i,s_{i+1},\dots,s_n)$.*
- *$p_i(s_{-i},\_\ ):V_i\to\mathbb{R}$ has continuous lateral partial derivatives*

$$\frac{\partial {p_i}^+}{\partial x_i^j}(s_{-i},\_\ ),\frac{\partial {p_i}^-}{\partial x_i^j}(s_{-i},\_\ ):V_i\to\mathbb{R}$$

  $j=1,\dots,m_i$ and

-
$$\frac{\partial {p_i}^+}{\partial x_i^j}(s_{-i},x_i)\leq\frac{\partial {p_i}^-}{\partial x_i^j}(s_{-i},x_i),\ \forall\ x_i\in V_i$$

With these assumptions, if $\chi(S_i)\neq 0$ for $i=1,2,\dots,n$ then $p_1,p_2,\dots,p_n$ have at least one w.l.e..

The proof of Theorem 3.1 is an application of a fixed point theorem of multivalued maps.

3.1 The Lefschetz Fixed Point Theorem for Admissible Multivalued Mappings

The spaces considered here are metric. Also, we are considering the Čech homology functor with compact carriers and with coefficients in $\mathbb{Q}$.

A proper map $f : X \to Y$ is a map such that, for all $K \subset X$ compact, $f^{-1}(K)$ is compact.

A compact space X is called acyclic if $H_0(X) = \mathbb{Q}$ and $H_q(X) = 0$ for $q > 0$.

A map $p : (X, X_0) \to (Y, Y_0)$ is called a Vietoris map if $p : X \to Y$ is proper, $p^{-1}(Y_0) = X_0$ and $p^{-1}(y)$ is acyclic, for every $y \in Y$. Symbol: $p : (X, X_0) \Rightarrow (Y, Y_0)$.

Theorem 3.2 (Vietoris Mapping Theorem)**.** *If $p : (X, X_0) \Rightarrow (Y, Y_0)$ is a Vietoris map then $p_* : H_*(X, X_0) \to H_*(Y, Y_0)$ is an isomorphism.*

Let X and Y be two spaces and assume that for each point $x \in X$ a nonempty closed subset $\varphi(x)$ of Y is given; in this case, we say that φ is a multivalued map from X into Y and we write $\varphi : X \multimap Y$.

A multivalued map $\varphi : X \multimap Y$ is called upper semicontinuous (u.s.c.) if for every open subset U of Y the set $\varphi^{-1}(U) = \{x \in X \mid \varphi(x) \subset U\}$ is an open subset of X.

An u.s.c. multivalued map $\varphi : X \multimap Y$ is called acyclic if for every $x \in X$ the set $\varphi(x)$ is an acyclic subset of Y.

An u.s.c. multivalued map $\varphi : X \multimap Y$ is called admissible if there exists a space Γ and mappings $p : \Gamma \Rightarrow X$, $q : \Gamma \to Y$ such that:

- p is a Vietoris map,
- $q(p^{-1}(x)) \subset \varphi(x)$, for every $x \in X$.

(p, q) is called a selected pair of φ (written $(p, q) \subset \varphi$).

Let $\varphi : X \multimap Y$ be an admissible multivalued map. The set $\{\varphi\}_*$ of linear induced mappings is defined by

$$\{\varphi\}_* = \{q_* p_*^{-1} : H_*(X) \to H_*(Y) \mid (p, q) \subset \varphi\}$$

Two admissible multivalued maps $\varphi, \psi : X \multimap Y$ are called homotopic (written $\varphi \sim \psi$) if there exists an admissible multivalued map $\chi : X \times [0,1]$ such that:

$$\chi(x,0) \subset \varphi(x) \quad \text{and} \quad \chi(x,1) \subset \psi(x) \quad \text{for every } x \in X$$

Theorem 3.3 ([5], Theorem (40.11))**.** *Let $\varphi : X \multimap Y$ be two admissible multivalued maps. Then $\varphi \sim \psi$ implies that there exists selected pairs $(p,q) \subset \varphi$ and $(\bar{p}, \bar{q}) \subset \psi$ such that*

$$q_* p_*^{-1} = \bar{q}_* \bar{p}_*^{-1}$$

Let X be a compact ANR and let $\varphi : X \multimap X$ be an admissible multivalued map. Then, it is well defined the Lefschetz set $\Lambda(\varphi)$ of φ by putting

$$\Lambda(\varphi) = \{\Lambda(q_* p_*^{-1}) = \sum_i (-1)^i \mathrm{trace}_i(q_* p_*^{-1}) \mid (p,q) \subset \varphi\}$$

Theorem 3.4 (Lefschetz fixed point theorem for admissible multivalued mappings)**.** *Let X be a compact ANR and $\varphi : X \multimap X$ be a compact admissible multivalued map. If $\Lambda(\varphi) \neq \{0\}$ then $Fix(\varphi) \neq \emptyset$.*

3.2 Proof of Theorem 3.1

In order to prove Theorem 3.1 we will define an admissible multivalued map $F : S \multimap S$ and we will prove that if $\tilde{s} \in F(\tilde{s})$ then $\tilde{s}$ is an w.l.e. for $p_1, \ldots, p_n$. The conclusion of the proof will follow from the Lefschetz fixed point theorem for admissible multivalued mappings. First, we need the following lemma.

Lemma 1. *Let X be a compact subset of $\mathbb{R}^m$ and let V be an open neighborhood of X in $\mathbb{R}^m$. Then, given a multivalued map $\varphi : X \multimap \mathbb{R}^m$ u.s.c. with compact values, there exists $t_1 > 0$ such that $x + tv \in V$ for all $x \in X$, $v \in \varphi(x)$ and $t \in [0, t_1]$.*

Proof. Let $\varphi : X \multimap \mathbb{R}^m$ be an u.s.c. multivalued map with compact values. If $\varphi(x) = \{0\}$ for every $x \in X$, there is nothing to prove. Suppose

$\varphi(x) \neq \{0\}$ for some $x \in X$. Since X is compact and φ is u.s.c. with compact values, the image $\varphi(X) = \bigcup_{x \in X} \varphi(x)$ is also compact. Then, the real number $u = \max_{v \in \varphi(X)} \{\|v\|\}$ is a finite positive number. For every $x \in X$, there is $\epsilon_x > 0$ such that $B(x, \epsilon_x) \subset V$. Since X is compact, we obtain a finite open subcover $\left\{B\left(x_i, \frac{\epsilon_{x_i}}{4}\right)\right\}_{i=1}^{l}$ with

$$X \subset \bigcup_{i=1}^{l} B\left(x_i, \frac{\epsilon_{x_i}}{4}\right) \subset \bigcup_{i=1}^{l} B(x_i, \epsilon_{x_i}) \subset V.$$

Let $\epsilon = \min_{1 \leq i \leq l} \left\{\frac{\epsilon_{x_i}}{4}\right\}$ and $t_1 = \frac{\epsilon}{u}$. Thus, $x + tv \in V$ for all $x \in X$, $v \in \varphi(x)$ and $t \in [0, t_1]$. In fact, given $x \in X$, we have $x \in B\left(x_i, \frac{\epsilon_{x_i}}{4}\right)$ for some x_i. If $v = 0$ the conclusion is obvious. If $v \neq 0$ then, given $t \in [0, t_1]$, we have

$$t \leq t_1 = \frac{\epsilon}{u} \leq \frac{\epsilon_{x_i}}{4u} \leq \frac{\epsilon_{x_i}}{4\|v\|}.$$

It follows that

$$\|x + tv - x_i\| \leq \|x - x_i\| + t\|v\| \leq \frac{\epsilon_{x_i}}{4} + \frac{\epsilon_{x_i}}{4\|v\|}\|v\| = \frac{\epsilon_{x_i}}{2} < \epsilon_{x_i}.$$

Therefore, $x + tv \in B(x_i, \epsilon_{x_i}) \subset V$.

Hence, for all $x \in X$, $v \in \varphi(x)$ and $t \in [0, t_1]$. □

Proof of Theorem 3.1. Since $S_1 \subset \mathbb{R}^{m_1}, \ldots, S_n \subset \mathbb{R}^{m_n}$ are compact ENR's with the p.c.r., the product $S = S_1 \times \cdots \times S_n \subset \mathbb{R}^m$ is also a space with the p.c.r, $m = m_1 + \cdots + m_n$. Thus, let $r : V \to S$ be a convenient retraction.

Let $\varphi : S \multimap \mathbb{R}^m$ be the multivalued map defined by

$$\varphi(s) = \varphi_1(s) \times \cdots \times \varphi_n(s)$$

where $\varphi_i(s) = DSp_i(s_{-i}, s_i)$.

>From Lemma 1, there exists $t_1 > 0$ such that $s + tv \in V$ for all $s \in S$, $t \in [0, t_1]$ and $v \in V(s)$.

Finally, we define $F : S \multimap S$ by

$$F(s) = \{r(s + t_1 v) \mid v \in \varphi(s)\}.$$

As defined, F is a compact admissible multivalued map. Moreover, F is homotopic to the identity map via homotopy $\psi : S \times [0, t_1] \to S$ given by $\psi(s,t) = \{r(s+tv) \mid v \in \varphi(s)\}$. Thus, by Theorema 3.3, there exists a selected pair $(p,q) \subset F$ such that

$$\Lambda(q_* p_*^{-1}) = \Lambda(id_S) = \chi(S) = \chi(S_1) \cdots \chi(S_n).$$

If $\chi(S_i) \neq 0$, $i = 1, \ldots, n$, then $\Lambda(F) \neq \{0\}$. It follows, from Theorem 3.4, that F has a fixed point, ie, a point $\tilde{s} \in S$ such that $\tilde{s} \in F(\tilde{s})$. We affirm that a such fixed point $\tilde{s}$ is a w.l.e. for $p_1, \ldots, p_n$. In fact, if $\tilde{s} \in F(\tilde{s})$ then $\tilde{s} = r(\tilde{s} + t_1 v)$ for some $v \in \varphi(\tilde{s})$. Since r is a convenient retraction, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|x - r(\tilde{s} + t_1 v)\| = \|x - \tilde{s}\| < \delta$$

implies that

$$\begin{aligned} \langle \tilde{s} + t_1 v - r(\tilde{s} + t_1 v), x - r(\tilde{s} + t_1 v)\rangle &= t_1 \langle v, x - \tilde{s}\rangle \\ &\leq \frac{t_1 \varepsilon}{2}\|x - \tilde{s}\|. \end{aligned}$$

Moreover, from the definition of φ, we can assume that if $\|\tilde{s} - s\| < \delta$ then

$$p_i(\tilde{s}_1, \ldots, \tilde{s}_{i-1}, s_i, \tilde{s}_{i+1}, \ldots, \tilde{s}_n) \leq p_i(\tilde{s}) + \langle v_i, s_i - \tilde{s}_i\rangle + \frac{\varepsilon}{2}\|s_i - \tilde{s}_i\|,$$

$1 \leq i \leq n$. It follows that, if $s \in S$ and $\|s - \tilde{s}\| < \delta$ then

$$p_i(\tilde{s}_1, \ldots, \tilde{s}_{i-1}, s_i, \tilde{s}_{i+1}, \ldots, \tilde{s}_n) \leq p_i(\tilde{s}) + \varepsilon\|s_i - \tilde{s}_i\|,$$

$1 \leq i \leq n$.

Hence, $\tilde{s}$ is a w.l.e. for $p_1, \ldots, p_n$.

$\square$

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Semi-free R^1 action and Bott map

1 Introduction

Let M^n be a compact closed manifold of dimension at least 3. We study the R^1-Bott functions on M^n. Separately investigated R^1-invariant Bott functions on M^{2n} with a semi-free circle action which has finitely many fixed points. The aim of this paper is to find exact values of minimal numbers of singular circles of some indices of R^1-invariant Bott functions on M^{2n}.

Closely related to R^1-Bott function on a manifold M^n is a more flexible object, the decomposition of round handle of M^n. In its turn, to study the round handles decomposition of M^n we use a diagram, i.e. a graph which carries the information about the handles.

2 R^1-Bott maps

Let M^n be a smooth manifold and $f : M^n \to \mathbf{R^1}$ smooth function or $f : M^n \to \mathbb{R}^1$ non-homotopy to zero a smooth map. Suppose that $x \in M^n$ one of its critical points of f. In neighborhood U of critical point x in both cases the map f can be viewed as a function with values in $\mathbf{R}$. Consider the Hessian $\Gamma_x(f) : T_x \times T_x \to \mathbf{R}$ at this point. Recall that the index of the Hessian is called the maximum dimension of T_x, where $\Gamma_x(f)$ is negative definite. The index of $\Gamma_x(f)$ is called the index of the critical point x, and the corank of $\Gamma_x(f)$ is called the corank of x. Suppose that the set of critical points of f forms a disjoint union of smooth submanifolds K^i_j whose their dimensions do not exceed $n - 1$. A connected critical submanifold $K^{i_0}_{j_0}$ is called **non-degenerate** if the

Hessian is non-degenerate on subspaces orthogonal to $K^{i_0}_{j_0}$ (i.e. has corank equal to $n - i_0$) at each point $x \in K^{i_0}_{j_0}$.

Definition 2.1. *A mapping $f : M^n \to \mathbb{R}^1$ is called a Bott map if all of its critical points form nondegenerate critical submanifolds which do not intersect the boundary of M^n.*

Consider the following important example of Bott map:

Definition 2.2. *A mapping $f : M^n \to \mathbb{R}^1$ is called an R^1-Bott map if all of its critical points form nondegenerate critical circles.*

Note that an R^1-Bott map do not exist on any smooth manifold (see Theorem 2.3).

Theorem 2.1. *Let M^n be a smooth closed manifold and suppose that on M^n there is R^1- Bott map $f : M^n \to \mathbb{R}^1$. Denote by $\gamma \subset M^n$ its critical circle and let $f(\gamma) = a$. Then there is interval $(a - \varepsilon, a + \varepsilon) \subset \mathbb{R}^1$ and a system of coordinates in a neighborhood of γ of one of the following types:*

1) *Trivial $\nu : S^1 \times D^{n-1}(\varepsilon) \to M^n$; where $D^{n-1}(\varepsilon)$, a disc of radius ε, $\nu(R^1 \times 0) = \gamma$, and $f(\nu(\theta, x)) = a - x_1^2 - ... - x_\lambda^2 + x_{\lambda+1}^2 + ... + x_{n-1}^2$, for $(\theta, x) \in S^1 \times D^{n-1}(\varepsilon)$.*

2) *Twisted $\tau : ([0,1] \times D^{n-1}(\varepsilon)/ \sim) \to M^n$, where τ is a smooth embedding such that $(\tau([0,1]) \times 0/ \sim) = \gamma$ and $f(\tau(t,x)) = a - x_1^2 - ... - x_\lambda^2 + x_{\lambda+1}^2 + ... + x_{n-1}^2$, for $(t,x) \in (\tau : [0,1] \times D^{n-1}(\varepsilon)/ \sim)$. Here $([0,1] \times D^{n-1}(\varepsilon)/ \sim)$ is diffeomorphic to $S^1 \times D^{n-1}(\varepsilon)$ by identifying $0 \times D^{n-1}(\varepsilon)$ and $1 \times D^{n-1}(\varepsilon)$ by the mapping: $(0, x_1, ..., x_\lambda, x_{\lambda+1}, ..., x_{n-1}) \leftrightarrow (1, -x_1, ..., x_\lambda, -x_{\lambda+1}, ..., x_{n-1})$.*

The number λ is called the index of the critical circle γ.

Let M^n be a smooth manifold, and $f : M^n \to R^1$ an R^1-Bott map. Each nice R^1-Bott map defines a filtration on manifold $M^n : \mathbb{M}_0(f) \subset \mathbb{M}_1(f) \subset ... \subset \mathbb{M}_{n-1}(f) \subset M^n$. The existence of a nice R^1-Bott map from manifold M^n into the circle is equivalent to existance of a R^1-round handle decomposition on the manifold M^n. We recall some necessary definitions.

Definition 2.3. *We define an n-dimensional round handle R_λ of index λ by $M_\lambda = M^1 \times D^\lambda \times D^{n-\lambda-1}$, where D^i is a disc of dimension i.*
Define twisted n-dimensional round handle TM_λ of index λ $(0 < \lambda < n-1)$ by $TM_\lambda = [0,1] \times D^\lambda \times D^{n-\lambda-1}/\sim$, where identification is given by the map: $(0, x_1, ..., x_\lambda, x_{\lambda+1}, ..., x_{n-1}) \leftrightarrow (1, -x_1, ..., x_\lambda, -x_{\lambda+1}, ..., x_{n-1})$.

Definition 2.4. *We say that the manifold M^n_λ is obtained from a smooth manifold M^n by attaching a round handle of index λ if*
$M^n_\lambda = M^n \bigcup_\varphi S^1 \times D^\lambda \times D^{n-\lambda-1}$, where $\varphi : R^1 \times \partial D^\lambda \times D^{n-\lambda-1} \longrightarrow \partial N^n$ is a smooth embedding.

Manifold M^n_λ is obtained from a smooth manifold M^n by gluing a twisted round handles of index λ, if $M^n_\lambda = N^n \bigcup_\varphi [0,1] \times D^\lambda \times D^{n-\lambda-1}/\sim$, where $\varphi : ([0,1] \times \partial D^\lambda \times D^{n-\lambda-1}/\sim) \rightarrow M^n$ is a smooth embedding.

Definition 2.5. *The M^1- round handle decomposition on the closed manifold M^n is called a filtration*

$$M^{n-1} \times [0,\varepsilon] \bigcup M^n_0(R) \subset M^n_1(R) \subset ... \subset M^n_{n-1}(R) = M^n,$$

where M^{n-1} is a closed submanifold of M^n, the manifold $M^n_i(R)$ obtained from the manifold $M^n_{i-1}(R)$ by gluing round and twisted round handles of index i .

In what follows we recall the relationship between S^1 and the decomposition by round handles ([11]).

Theorem 2.2. *Let M^n be a smooth closed manifold. The following two conditions are equivalent:*

1) *On the manifold M^n there is a nice R^1-Bott map with the critical circles $\gamma_1, ..., \gamma_k$ of index $\lambda_1, ..., \lambda_k$ with trivial coordinate systems and critical circles $\tilde{\gamma}_1, ..., \tilde{\gamma}_l$ of indices $\mu_1, ..., \mu_l$ with twisted coordinate systems.*

2) *Manifold M^n admits a decomposition by round handles consisting of round handles $R_{\lambda_1}, ..., R_{\lambda_k}$ of index $\lambda_1, ..., \lambda_k$ and of twisted round handles*
$TR_{\mu_1}, ..., TR_{\mu_l}$ of indices $\mu_1, ..., \mu_l$ so that the critical circle γ_i corresponds to a round handle R_{λ_i} $(1 \le i \le k)$, and the critical circle $\tilde{\gamma}_j$ corresponds to a twisted round handle TR_{μ_j} $(1 \le j \le l)$.

Thus each nice R^1-Bott map from manifold M^n into the $\mathbb{R}^1$ generates a round handle decomposition of M^n and vice versa.

We are interested in conditions when an R^1-Bott map on M^n has the property that all of its critical circles have trivial coordinate system. We recall the necessary facts from an [4].

Lemma 2.1. *Let M^n be a smooth closed manifold, $f : M^n \to \mathbb{R}^1$ an R^1-Bott map, and c its critical value. Suppose $\varepsilon > 0$, and that on the interval $[c-\varepsilon, c+\varepsilon]$ there are no other critical values. Assume that on the surface level $f^{-1}(c)$ there are critical circles $\gamma_1, ..., \gamma_k$ of indices $\lambda_1, ..., \lambda_k$ with trivial coordinate systems and there are critical circles $\tilde{\gamma}_1, ..., \tilde{\gamma}_l$ of indices $\mu_1, ..., \mu_l$ with twisted coordinate systems, then the homology groups $H_*(f^{-1}[c-\varepsilon, c+\varepsilon], f^{-1}(c-\varepsilon), \mathbf{Z})$ is generated exactly by the handles which correspond to the critical circles*
$\gamma_1, ..., \gamma_k, \tilde{\gamma}_1, ..., \tilde{\gamma}_l$. Each circle γ_i generates two subgroups that are isomorphic to $\mathbf{Z}$, a direct product of the homology group $H_{\lambda_i}(f^{-1}[c-\varepsilon, c+\varepsilon], f^{-1}(c-\varepsilon), \mathbf{Z})$, and the other in the homology group $H_{\lambda_{i+1}}(f^{-1}[c-\varepsilon, c+\varepsilon], f^{-1}(c-\varepsilon), \mathbf{Z})$. Each circle $\tilde{\gamma}_j$ generates a subgroup $\mathbf{Z}_2$ which is direct product in a group $H_{\mu_j}(f^{-1}[c-\varepsilon, c+\varepsilon], f^{-1}(c-\varepsilon), \mathbf{Z})$.

Corolary 2.1. *Let M^n be a smooth closed manifold, $f : M^n \to \mathbb{R}^1$ an S^1-Bott map, and $c_1, ..., c_k$ its critical values. Suppose $\varepsilon_i > 0 (1 \le i \le k)$ such that the interval $[c_i-\varepsilon_i, c_i+\varepsilon_i]$ has no other critical values. Then on a level surface $f^{-1}(c_i)$ there are only critical circles with trivial coordinate systems if and only if the nonzero homology groups $H_*(f^{-1}[c_i-\varepsilon_i, c_i+\varepsilon_i], f^{-1}(c_i-\varepsilon_i), \mathbf{Z})$ are free Abelian groups.*

Thus we have a homological criterion when R^1-Bott map do not have critical circle with twisted coordinate systems.

In the next section, we give another class of R^1-Bott map which do not possess the critical circle with twisted coordinate systems.

Definition 2.6. *Let M^n be a smooth closed manifold. The number*
$\chi_i(M^n) = \mu(H_i(M^n, \mathbf{Z})) - \mu(H_{i-1}(M^n, \mathbf{Z})) + \ldots + (-1)^{i+1}\mu(H_0(M^n, \mathbf{Z}))$ is called the i-th Euler characteristic of M^n,where $\mu(H)$ is a minimal number of generators H.

Definition 2.7. *A dimension λ of closed manifold M^n is called singular if $H_\lambda(M^n, \mathbf{Z})$ is a nonzero finite group distinct from $\mathbf{Z}_2 \oplus ... \oplus \mathbf{Z}_2$ and $\chi_{\lambda-1}(M^n) = \chi_{\lambda+1}(M^n) = 0$.*

Definition 2.8. *Let M^n be a smooth closed manifold. A round handle decomposition is called quasiminimal, if one of the following holds:*

1) *the number of round handles of index i equals to $\rho(\chi_i(M^n)) + \varepsilon_i$, where $\varepsilon_i = 0$,if dimension $i+1$ is nonsingular and $\varepsilon_i = 1$,if dimension $i+1$ is singular,*

2) *the number of round handles of index i equals to $\rho(\chi_i(M^n))$, if dimension $i+1$ is singular, then there is only one handle of index $i+2$.*

In both cases, the number of round handles of index $i+1$ equals to $\rho(\chi_{i+1}(M^n))$. A round handle decomposition is called minimal, if number of round handles of index i equals to $\rho(\chi_i(M^n))$ for all i.

Using the decomposition of manifold on handles and the diagram technique, we can easily prove the following fact [4].

Proposition 2.1. *Let M^n be a smooth closed simply connected manifold $(n > 5)$. Then M^n admits a* **quasiminimal** *decomposition into round handles. If manifold M^n have not singular dimensions, then M^n admits a* **minimal** *decomposition into round handles.*

Definition 2.9. *Let the manifold M^n admits R^1-Bott function, then* **R^1-Morse number** $M_i^{R^1}(M^n)$ **of index** *i is the minimum number of singular circles of index i taken over all R^1-Bott functions on M^n.*

Lemma 2.2. *Let on a closed manifold M^n exist a smoth function $f : M^n \to \mathbb{R}$ such that each connected component of the singular set Σ_f of f is either a nondegenerate critical point $p_i (i = 1, ..., k)$ or a non-degenerate critical circle $S_j^1 (j = 1, ..., l)$. Then the Euler characteristic of the manifold M^n is equal to $\chi(M^n) = \sum_{i=1}^k (-1)^{index(p_i)}$.*

Proof. It is known that for any Morse function on the manifold M^n $g : M^n \to \mathbb{R}$ with critical points $p_i (i = 1, ..., q)$ there is the formula $\chi(M^n) = \sum_{i=1}^q (-1)^{index(p_i)}$. By small perturbation of the function f any non-degenerate critical circle S_j^1 of index λ can be replaced by non-degenerate critical points of idexes λ and $\lambda + 1$ [1]. Therefore the contribution in the formula of Euler characteristic this critical points will not give and we obtain the desired formula. □

3 Manifolds with free R^1-action

Let on smooth manifold M^n there is smooth free circle action. Then of course the set M^n/S^1 is a manifold and natural projection $p : M^n \to$

M^n/S^1 is fibre bundle. Any smooth R^1-invariant map $f : M^n \to \mathbb{R}^1$ from the manifold M^n into the circle $\mathbb{R}^1$ is called an **R^1-invariant round Bott map** if each connected component of the singular set Σ_f is non-degenerate critical circle.

It is clear that if f be a R^1-invariant round Bott map from the manifold M^n then it projection $\pi_*(f) : M^n/S^1 \to \mathbb{R}^1$, is a Morse map. And conversaly, if $g : M^n/S^1 \to \mathbb{R}^1$ be a Morse map from the manifold M^n/S^1 then $\pi_*^{-1}(g) = g \circ \pi : M^n \to \mathbb{S}^1$ is R^1-invariant round Bott map from the manifold M^n. The critical point of the index λ of the map g correspond to critical circle of the index λ of the map $\pi_*^{-1}(g)$.

Definition 3.1. *Let on smooth manifold M^n there are smooth free circle action $\theta : M^n \times S^1 \to M^n$ and R^1-invariant round Bott map $f : M^n \to \mathbb{S}^1$. For the triple (M^n, θ, f)* **R^1-equivariant round Morse-Bott number of index** *i, $\mathfrak{M}_i^{eqS^1}(M^n, \theta, f)$ is the minimum number of singular circles of index i taken over all homotopic to f R^1-invariant round Bott map from M^n into $\mathbb{R}^1$.*

Definition 3.2. *Let on smooth manifold M^n there is Morse maps $f : M^n \to \mathbb{R}^1$. For the couple (M^n, f)* **Morse-Novikov number of index** *i, $\mathfrak{M}_i(M^n, f)$ is the minimum number of critical points of index i taken over all homotopic to f Morse maps from M^n into $\mathbb{R}^1$.*

It is clear that there is following fact.

Corolary 3.1. *Let on smooth manifold M^n there is smooth free circle action $\theta : M^n \times R^1 \to M^n$ and let $p : M^n \to M^n/R^1$ is natural projection. Suppose that $f : M^n/R^1 \to \mathbb{R}^1$ be a Morse map. Then $\mathfrak{M}_i^{eqR^1}(M^n, \theta, f \cdot p) = \mathfrak{M}_i(M^n/S^1, f)$.*

Definition 3.3. *Let on smooth manifold M^n there is smooth free circle action $\theta : M^n \times R^1 \to M^n$. Then this circle action is* **minimal** *if there exist R^1-invariant round Bott map $f : M^n \to \mathbb{R}^1$ such that $\mathfrak{M}_i^{eqR^1}(M^n, \theta, f) = \mathfrak{M}_i^{S^1}(M^n, f)$ for all i.*

Suppose that on smooth compact manifold $M^n (n > 6)$ there is smooth free circle action $\theta : M^n \times R^1 \to M^n$ and let $p : M^n \to M^n/R^1$ is natural projection. Suppose that $\pi_1(M^n) \approx \pi_1(M^n/R^1) \approx \mathbf{Z}$. Then from from results of Novikov [2] it follows that

$$\mathfrak{M}_i(M^n/R^1, f) = \mu(H_i(M^n/R^1, Z)) + \mu(TorsH_{i-1}(M^n/R^1, Z))$$

for any non-homotopy to zero Morse map $f : M^n/R^1 \to \mathbb{R}^1$. Therefore corollary 3.1 implies that $\mathfrak{M}_i^{eqR^1}(M^n, \theta, f \cdot p) = \mathfrak{M}_i^{R^1}(M^n, f)$

Theorem 3.1. *Let on smooth compact manifold $M^n(n > 6)$ there is smooth free circle action. Suppose that $\pi_1(M^n) \approx \pi_1(M^n/S^1) \approx \mathbf{Z}$. Then this circle action is minimal if and only if*

$$\mu(H_i(M^n/S^1, Z) + \mu(TorsH_{i-1}(M^n/S^1, Z) = \rho(\chi_i(M^n))$$

for all i.

Proof. **Necessary.** Suppose that on M^n there is minimal smooth free circle action. If $n > 6$ from results of Novikov [2] it follows that Morse number in dimension i of the manifold M^n/R^1 is equal
$\mathfrak{M}_i(M^n/S^1) = \mu(H_i(M^n/R^1, Z)) + \mu(TorsH_{i-1}(M^n/R^1, Z))$. There is equality
$\mathfrak{M}_i(M^n/S^1) = \mathfrak{M}_i^{eqR^1}(M^n)$. Because of the condition of minimal free circle action there is equality $\mathfrak{M}_i(M^n/R^1) = \mathfrak{M}_i^{eqR^1}(M^n) = \mathfrak{M}_i^{R^1}(M^n) = \rho(\chi_i(M^n))$.

Sufficiently. Consider on manifold M^n/R^1 Morse function with the number of critical points of index i equal
$M_i(M^n/R^1) = \mu(H_i(M^n/R^1, Z)) + \mu(TorsH_{i-1}(M^n/R^1, Z))$. By the construction and condition of the theorem we have the equalities
$M_i(M^n/S^1) = M_i^{eqR^1}(M^n) = \rho(\chi_i(M^n))$. But $M_i^{R^1}(M^n) = \rho(\chi_i(M^n))$ and therefore free action of R^1 is minimal. □

4 Manifolds with semi-free R^1-action

Let M^{2n} be a closed smooth manifold with semi-free R^1-action which has only isolated fixed points. It is known that every isolated fixed point p of a semi-free R^1-action has the following important property: near such a point the action is equivalent to a certain linear $S^1 = SO(2)$-action on $\mathbb{R}^{2n}$. More precisely, for every isolated fixed point p there exist an open invariant neighborhood U of p and a diffeomorphism h from U to an open unit disk D in $\mathbb{C}^n$ centered at origin such that h is conjugate to the given S^1-action on U to the S^1-action on $\mathbb{C}^n$ with weight $(1, \ldots, 1)$. We will use both complex, $(z_1, \ldots, z_n)$, and real coordinates $(x_1, y_1, \ldots, x_n, y_n)$ on $\mathbb{C}^n = \mathbb{R}^{2n}$ with $z_j = x_j + \sqrt{-1}y_j$. The pair (U, h) will be called a **standard chart** at the point p. Let $f : M^{2n} \to \mathbb{R}^1$ be a smooth R^1-invariant map from the manifold M^{2n} into the circle $\mathbb{R}^1$. Denote by Σ_f

the set of singular points of the map f. It is clear that the set of isolated singular points $\Sigma_f(p_j) \subset \Sigma_f$ of f coincides with the set of fixed points M^{R^1}.

For a nondegenerate critical point p_j there exist a standard chart (U_j, h_j) such that on U_j the map f is given by the following formula:

$$f = f(p) - |z_1|^2 - \ldots - |z_{\lambda_j}|^2 + |z_{\lambda_j+1}|^2 + \ldots + |z_n|^2.$$

Notice that the index of nondegenerate critical point p_j is always even.

Denote by $\Sigma_f(R^1)$ the set singular points of the function f that are disconnected union of circles. These circles will be called singular.

A circle $s \in \Sigma f_(R^1)$ is called nondegenerate if there is an R^1-invariant neighborhood U of s on which R^1 acts freely and such that the point $\pi(s)$ is nondegenerate for the function $\pi_*(f) : U/R^1 \to \mathbb{R}$, induced on U/R^1 by the natural map $\pi : U \to U/R^1$. An invariant version of Morse lemma says that there exist an R^1-invariant neighborhood U of the circle s and coordinates $(x_1, \ldots, x_{2n-1})$ on U/R^1 such that the function $\pi_*(f)$ has the following presentation:

$$\pi_*(f) = \pi_*(f(\pi(s))) - x_1^2 - \ldots - x_\lambda^2 + x_{\lambda+1}^2 + \ldots + x_{2n-1}^2.$$

By definition λ is the **index** of singular circle s.

Definition 4.1. *A smooth S^1-invariant function $f : M^{2n} \to \mathbb{R}$ on a manifold M^{2n} with a semi-free circle action which has isolated fixed points is called : R^1_*-Bott function if each connected component of the singular set Σ_f is either a nondegenerate fixed point or a nondegenerate critical circle.*

Theorem 4.1. *Assume that M^{2n} is the closed manifold with a smooth semi-free circle action which has isolated fixed points $p_1, \ldots, p_k$. Let for any fixed point p_j consider standard chart (U_j, h_j) and function*

$$f_j = f_j(p_i) - |z_1|^2 - \ldots - |z_{\lambda_j}|^2 + |z_{\lambda_j+1}|^2 + \ldots + |z_n|^2$$

on U_j, where λ_j is an ***arbitrary integer*** *from $0, 1, \ldots, n$.*

Then there exist an R^1-invariant R^1_*-Bott function f on M^{2n} such that $f = f_j$ on U_j.

Proof. Consider on U_j the function f_j. Let $\pi_*(f_j) : U_j/S^1 \to \mathbb{R}$, continuos function induced on U_j/R^1 by the natural map $\pi : U_j \to U_j/R^1$. It is clear that function $\pi_*(f_j)$ is smooth on manifold $(U_j \setminus$

$p_j)/R^1$. Denote by g smooth extension functions $\pi_*(f_j)$ on M^{2n}/R^1. By small deformation of the function g, that is fixed on U_j/R^1, we shall find function g_1 on M^{2n}/R^1 such that g_1 equal $\pi_*(f_j)$ on U_j/R^1 and g_1 have only non-degenerate critical points on $M^{2n} \setminus \bigcup(U_j/R^1)$. Then the function $f = g_1 \circ p$ satisfy conditions of the theorem. □

Theorem 4.2. *The number of fixed points of any smooth semi-free circle action on M^{2n} with isolated fixed points is always even and equal to the Euler characteristic of the manifold M^{2n}.*

$f_1 = f_1(p_1) + |z_1|^2 + \ldots + |z_n|^2$ on U_1 and $f_j = f_j(p_i) - |z_1|^2 - \ldots - |z_n|^2$

on U_j $(2 \leq j \leq l)$ and extend such functions to S^1-invariant Bott function f on manifold $M^{2n} \setminus U_1 \bigcup U_2 \bigcup ... \bigcup U_l$. We suppose that U_j is diffeomorfic to open disk D^{2n} for any j. Consider manifold $V^{2n} = W^{2n} \setminus \bigcup U_j$. The boudary of manifod V^{2n} is disconnected union of spheres S^{2n-1}. By construction of manifold V^{2n} there is free cirle action. The boundary of the manifold V^{2n}/S^1 is disconnected union of complex projective spaces $\mathbb{CP}^{n-1}$. If the number of the boundary components of the manifold V^{2n}/S^1 is odd then we glue pairwise boundary components and obtain compact smoth manifold with with boundary $\mathbb{CP}^{n-1}$. From the well known fact that the manifold $\mathbb{CP}^{n-1}$ is non-cobordant to zero it follows that the number of fixed points of any smooth semi-free circle action on M^{2n} with isolated fixed points is even. The value of the Euler characteristic $\chi(M^{2n}) = 2k$ is follow from Lemma 3.4.□

Definition 4.2. *Let f be an R^1-invariant S^1_*-Bott function for smooth semi-free circle action with isolated fixed points $p_1, \ldots, p_{2k}$ on a closed manifold M^{2n}. Denote by λ_j the index of a critical point p_j of the function f. The* **state** *of the function f is the collection of numbers $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2k})$, which we will be denoted by $St_f(\Lambda)$. It is clear that all numbers λ_j are even and $(0 \leq \lambda_j \leq 2n)$.*

Remark 4.1. *It follows from Theorem 4.2 that for every smooth semi-free circle action on a closed manifold M^{2n} with isolated fixed points $p_1, \ldots, p_{2k}$ and any collection even numbers $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2k})$, such that $0 \leq \lambda_j \leq 2n$ there exists an R^1-invariant R^1_*-Bott functions f on M^{2n} with state $St_f(\Lambda)$.*

Definition 4.3. *Let M^{2n} be a closed smooth manifold with smooth semi-free circle action which has finitely many fixed points $p_1, \ldots, p_{2k}$. Fix any collection even numbers $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2k})$, such that $0 \leq \lambda_j \leq 2n$.*

The R^1-Morse number $\mathcal{M}_i^{R^1}(M^{2n}, St(\Lambda))$ of index i is the minimum numbers of singular circles of index i taken over all R^1-invariant R^1_*-Bott functions f on M^{2n} with state $St_f(\Lambda)$.

There is an unsolved problem: for a manifold M^{2n} with a semi-free circle action which has finitely many fixed points **find exact values** of numbers $\mathcal{M}_i^{R^1}(M^{2n}, St(\Lambda))$.

5 About R^1-equivariant Morse numbers $\mathcal{M}_i^{R^1}(M^{2n}, St(\Lambda))$

Let M^{2n} be a compact closed manifold of dimension with semi-free circle action which has finite many fixed points $p_1, , ..., p_{2k}$. Denote by $\pi : M^{2n} \to M^{2n}/R^1$ canonical map. The set M^{2n}/R^1 is manifold with singular points $\pi(p_1), , ..., \pi(p_{2k})$. It is clear that neighborhood of any singular point is cone over $\mathbb{CP}^{n-1}$. If $f : M^{2n} \to \mathbb{R}$ be a smooth R^1-invariant R^1_*-Bott function on the manifold M^{2n}, then $\pi_*(f) : M^{2n}/R^1 \to \mathbb{R}$ is continuos function such that on smooth noncompact manifold $N^{2n-1} = M^{2n}/R^1 \setminus \bigcup_{j=1}^{2k} \pi(p_j)$ it is Morse function.

Choose an invariant neighborhood U_i of the point p_j diffeomorphic to the open unit disc $D^{2n} \subset \mathbb{C}^n$ and set $U = \bigcup_{j=1}^{2k} U_j$. Consider compact manifold $V^{2n-1} = (M^{2n} \setminus U)/R^1$, its boundary is a disconnected union of complex projective spaces $\partial V^{2n-1} = \mathbb{CP}_1^{n-1} \cup \ldots \cup \mathbb{CP}_{2k}^{n-1}$. It is clear that manifold $V^{2n-1} \setminus \partial V^{2n-1}$ and manifold N^{2n-1} are diffeomorphic. We use a manifold V^{2n-1} for the study of R^1-invariant R^1_*-Bott functions on the manifold M^{2n} with states $St(\Lambda) = (0, \ldots, 0, 2n, \ldots, 2n)$. Let $\partial_0 V^{2n-1}$ be a part of boundary of V^{2n-1} consist from r component $\mathbb{C}P^{2n-2}$ $(2k-1 \geq r \geq 1)$, and $\partial_1 V^{2n-1} = \partial V^{2n-1} \setminus \partial_0 V^{2n-1}$. On the manifold with boundary V^{2n-1} constructed Morse function $f : V \to [0,1]$, such that $f^{-1}(0) = \partial_0 V^{2n-1}$ and $f^{-1}(1) = \partial_1 V^{2n}$. Using the function f we constructed on the manifold M^{2n} R^1-equivariant R^1_*-Bott function F with the state $St(0, \ldots, 0, 2n, \ldots, 2n)$, such that restriction $\pi_*(F)$ on V coinside with f. Therefore Morse number of index i $M_i(V^{2n-1}, \partial_0 V^{2n-1})$ of manifold with boundary V^{2n-1} is equal $\mathcal{M}_i^{S^1}(M^{2n}, St(0, \ldots, 0, 2n, \ldots, 2n)$.

Theorem 5.1. *Let M^{2n} $(2n > 8)$ be a closed smooth manifold admits a smooth semi-free circle action with isolated fixed points $p_1, \ldots, p_{2k}$. Then* *for the manifold M^{2n} with the state $St(\Lambda) = (0, \ldots, 0, 2n, \ldots, 2n)$*

$$\mathcal{M}_i^{R^1}(M^{2n}, St(\Lambda) = \mathbb{D}^i(V^{2n-1}, \partial_0 V^{2n-1}) + \widehat{S}^i_{(2)}(V^{2n-1}, \partial_0 V^{2n-1}) +$$

$$+ \widehat{S}^{i+1}_{(2)}(V^{2n-1}, \partial_0 V^{2n-1}) + dim_{N(Z[\pi])}(H^i_{(2)}(V^{2n-1}, \partial_0 V^{2n-1}))$$

for $3 \leq i \leq 2n-4$.

Proof. Choose an invariant neighborhood U_i of the point p_i diffeomorphic to the unit disc $D^{2n} \subset \mathbb{C}^n$ and set $U = \bigcup_i U_i$. Let f_i be a function on U_i equal

$$f_i = |z_1|^2 + \ldots + |z_n|^2, \text{ and } f_j \text{ on } U_j \text{ equal } f_j = 1 - |z_1|^2 - \ldots - |z_n|^2,$$

for $i = 1, ..., r,\ j = r+1, ..., 2k-r$. Consider the manifold $V^{2n} = (M^{2n} \setminus U)/R^1$. It is clear that its boundary is a disconnected union of complex projective spaces $\partial V^{2n} = \mathbb{C}P_1^{2n-2} \cup \ldots \cup \mathbb{C}P_{2k}^{2n-2}$.

Let $\partial_0 V^{2n}$ be a part of boundary of V^{2n} consist from r component $\mathbb{C}P^{2n-2}$, that corespondent U_i and $\partial_1 V^{2n}$ be a part of boundary consist from component $\mathbb{C}P^{2n-2}$, that corespondent U_j. On manifold $V^{2n} = (M^{2n} \setminus U)/R^1$ constructed Morse function $f : V \to [0,1]$, such that $f^{-1}(0) = \partial_0 V^{2n}$ and $f^{-1}(1) = \partial_1 V^{2n}$. Using the function f we constructed on manifold M^{2n} S^1-equivariant S^1_*-Bott function F with the state $St(\Lambda) = (0, \ldots, 0, 2n, \ldots, 2n)$, such that restriction F on U_i coinside with f_i, restriction F on U_j coinside with f_j and restriction $\pi_*(F)$ on V coinside with f. Therefore Morse number of cobordism V equal $\mathcal{M}^{\lambda}_{R^1}(M^{2n}, St(\Lambda))$ In the paper [12] there is value of Morse number of a cobordism. $\square$

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