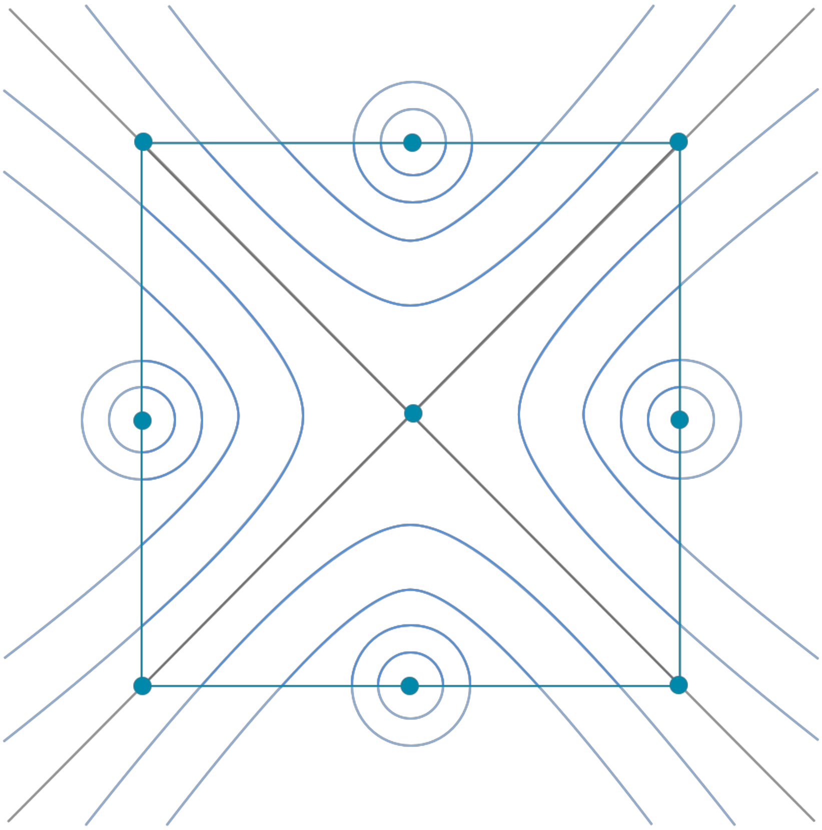
International Conference

# Morse theory and its applications



dedicated to the memory and 70th anniversary
of Volodymyr Sharko (25.09.1949 — 07.10.2014)

Institute of Mathematics of National Academy of
Sciences of Ukraine, Kyiv, Ukraine

September 25—28, 2019

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**Volodymyr Vasylyovych Sharko**
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The conference is dedicated to the memory and 70th anniversary of the outstanding topologist, Corresponding Member of National Academy of Sciences **Volodymyr Vasylyovych Sharko** (25.09.1949-07.10.2014) to commemorate his contributions to the Morse theory, $K$-theory, $L^2$-theory, homological algebra, low-dimensional topology and dynamical systems.

## List of topics

- Morse theory
- Low dimensional topology
- K-theory
- $L^2$-theory
- Dynamical systems
- Geometric methods of analysis

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# The continuity of Darboux injections between manifolds

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We shall discuss a problem of Willie Wong who asked on Mathoverflow if every bijective Darboux map $f : X \to Y$ between Eculidean spaces (more generally, between manifolds) is a homeomorphism. A function between topological spaces is *Darboux* if the image of any connected subset is connected. We prove that an injective Darboux map $f : X \to Y$ between connected metrizable spaces $X, Y$ is continuous if one of the following conditions is satisfied:

i) $Y$ is a 1-manifold and $X$ is compact;
ii) $Y$ is a 2-manifold and $X$ is a closed 2-manifold;
iii) $Y$ is a 3-manifold and $X$ is a rational homology 3-sphere.

More details can be found in the preprint https://arxiv.org/abs/1809.00401.

# Singularity of control in a model of acquired chemotherapy resistance

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This study investigates how optimal control theory may be used to delay the onset of chemotherapy resistance in tumours. We study a two-compartment model of drug-resistant tumour growth under angiogenic signalling. An optimal control problem with simple tumour dynamics and an objective functional explicitly penalising drug resistant tumour phenotype is formulated. Global existence and positivity of solutions, bifurcations (including bistability and hysteresis) with respect to the chemotherapy dose are studied. Two optimisation problems are then considered. In the first problem a constant, indefinite chemotherapy dose is optimised to maximise the time needed for the tumour to reach a critical (fatal) volume. In the second problem, an optimal dosage over a short, 30-day time period, is found. It is concluded that, after an initial full dose interval, an administration of intermediate dose is optimal over a broad range of parameters.

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# The topology of codimension one foliations with leaves of nonnegative Ricci curvature

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In this talk we shall discuss the topology of closed riemannian manifolds that admit codimension one foliations with leaves of nonnegative Ricci curvature. The following theorem is a foliated analogue of the famous Gheeger-Gromoll results about the structure of manifolds of nonnegative Ricci curvature [1].

**Theorem 1.** *Let $\mathcal{F}$ be a $C^\infty$-smooth codimension one transversally oriented foliation on closed oriented riemannian manifold $M^n$ with leaves of nonnegative Ricci curvature in the induced metric. Then*

- *the fundamental group of leaves of the foliation $\mathcal{F}$ is finitely generated and virtually abellian.*
- *the fundamental group $\pi_1(M^n)$ is virtually polycyclic;*
- *the foliation $\mathcal{F}$ is flat (i.e. the leaves of $\mathcal{F}$ are flat in the induced metric) iff $M^n$ is $K(\pi, 1)$-manifold.*

**Remark 2.** The first statement of the theorem is a positive answer to the Milnor conjecture for the leaves of codimension one foliations with induced metric of nonnegative Ricci curvature [2]. Recall that the Milnor conjecture claims that the fundamental group of a complete riemannian manifold of nonnegative Ricci curvature is finitely generated.

## References

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# Equilibrium positions of nonlinear differential-algebraic systems

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In this note we study particular deformations called ”CB-deformations” of a codimension 1 foliation into contact structures, in order to prove first that: if $M$ is a closed 3-manifold diffeomorphic to a quotient of the Lie group $G$ under a discrete subgroup $\Gamma$ acting by left multiplication, where $G$ is $\widetilde{SL_2}$ (the universal cover of $PSL_2\mathbb{R}$) or $\widetilde{E_2}$ (the universal cover of the group of orientation preserving isometries of the Euclidean plane), then there is on M, a codimension 1-foliation which is CB-deformable into con-tact structures. Secondly with theses deformations, which are more general than the linear one introduced by Eliashberg and Thurston [1], we can prove also that every $K$-contact structure on a $2n+1$-dimensional closed oriented manifold $M$ such that $dim(H^1(M)) > 0$, converges into a codimension 1-foliation. This last one can be view as a generalization of a theorem of Etnyre [2] on $2n+1$-dimensional closed $K$-contact manifolds.

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# Equilibrium positions of nonlinear differential-algebraic systems

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We investigate the problem of constructing solutions $z(t) \in \mathbb{C}^1[a,b]$ of the nonlinear differential algebraic system [1, 2, 3]

$$A(t)z'(t) = B(t)z(t) + f(t) + Z(z,t). \tag{1}$$

Here $A(t),\ B(t) \in \mathbb{C}_{m\times n}[a,b]$ is a continuous matrices, $f(t) \in \mathbb{C}[a,b]$ is a continuous vector. We consider a nonlinear function $Z(z,t)$ that assume twice continuously differentiable by $z$ in a certain region $\Omega \subseteq \mathbb{R}^n$ and continuous in $t \in [a,b]$. We call the equilibrium position of the system (1) a function $z(t) \in \mathbb{C}^1[a,b]$, that satisfies two conditions $A(t)z' = 0,\ B(t)z + f(t) + Z(z,t) = 0$. In the simplest case, under the condition $B(t) \equiv B,\ f(t) \equiv f -$ const, $Z(z,t) \equiv Z(z)$, the equilibrium position $z(t) \equiv z -$ const of a nonlinear differential-algebraic system (1) defines the equation

$$\varphi(z) := B\,z + f + Z(z) = 0. \tag{2}$$

To solve the equation (2), we apply the Newton method [4, 5].

**Lemma 1.** *Assume that the following conditions are satisfied for the equation (2):*

i) *The nonlinear vector function $\varphi(z)$ in the neighborhood $\Omega$ of the point $z_0$, has the root $z^* \in \mathbb{R}^n$.*

ii) *In the indicated neighborhood of the zeroth approximation $z_0$ the inequalities*

$$\left|\left|J_k^+\right|\right| \leq \sigma_1(k), \left|\left|d^2\varphi(\xi_k\,; z^* - z_k)\right|\right| \leq \sigma_2(k)\cdot||z^* - z_k||, \quad \theta := \sup_{k\in N}\left\{\frac{\sigma_1(k)\sigma_2(k)}{2}\right\}$$

*are satisfied.*

*Then under the conditions*

$$P_{J_k^*} = 0, \qquad J_k := \varphi'(z_k) \in \mathbb{R}^{m\times n}, \qquad \theta\cdot|z^* - z_0| < 1$$

*an iterative scheme $z_{k+1} = z_k - J_k^+\,\varphi(z_k)$ is applicable to find the solution $z^*$ of the equation (2). The vector function $z^*$ is the equilibrium position of the differential algebraic system (1).*

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# Distinsguishing Legendrian and transverse knots


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The talk is based on joint works (recent and in progress) with Maxim Prasolov and Vladimir Shastin.

A smooth knot (or link) $K$ in the three-space $\mathbb{R}^3$ is called *Legendrian* if the restriction of the 1-form $\alpha = x\,dy + dz$ on $K$ vanishes, where $x, y, z$ are the standard coordinates in $\mathbb{R}^3$. If $\alpha|_K$ is everywhere non-vanishing on $K$, then $K$ is called *transverse.*

Classification of Legendrian and transverse knots up to respectively Legendrian and transverse isotopy is an important unsolved problem of contact topology. A number of useful invariants have been constructed in the literature, but there are still small complexity examples in which the existing methods do not suffice to decide whether or not the given Legendrain (or transverse) knots are equivalent.

We propose a totally new approach to the equivalence problem for Legendrian and transverse knots, which allows to practically distinguish between non-equivalent Legendrain (or transverse) knots in small complexity cases, and gives rise to a complete algorithmic solution in the general case.

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# Lyusternik–Schnirelmann Theorem for $C^1$-functions on Fréchet spaces

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The Lusternik-Schnirelmann category $\mathrm{Cat}_X A$ of a subset $A$ of a topological space $X$ is the minimal number of closed sets that cover $A$ and each of which is contractible to a point in $X$. If $\mathrm{Cat}_X A$ is not finite, we write $\mathrm{Cat}_X A = \infty$.

Let $(E, \| \cdot \|^n)$ be a Fréchet space, $\varphi : E \to R$ a $C^1$-function. Suppose $\mathrm{Cr}(\varphi)$ is the set of critical points of $\varphi$ and for all $c \in \mathbb{R}$

$$\mathrm{Cr}(\varphi, c) = \{x \in E : \varphi'(x) = 0, \varphi(x) = c\},$$

$$E^c = \{x \in E : \varphi(X) \leqq c\}.$$

Let Co(E) be the set of compact subsets of $E$. Define the sets

$$\mathcal{A}_i = \{A \subset E : A \in \mathrm{Co(E)}, \mathrm{Cat}_E A \geqq i, i \in \mathbb{N}\},$$

and the numbers

$$\mu_i = \inf_{A \in \mathcal{A}_i} \sup_{x \in A} \varphi(x).$$

Assume $e \in E$, we define

$$\Xi\varphi(e) = \inf \Big\{ d\varphi(e, h) : h \in E, \| h \|^n = 1, \forall n \in \mathbb{N} \Big\},$$

where $d$ is the derivative of $\varphi$ at $e$ in the direction of $h$.

**Definition 1.** Let $\varphi : E \to \mathbb{R}$ be a $C^1$-functional, we say that $\varphi$ satisfies the Palais-Smale condition at the level $c$ if any sequence $x_i \in E$ such that $\varphi(x_i) \to c$ and $\Xi\varphi(x_i) \to 0$ has the convergent sub-sequence.

**Theorem 2.** *Suppose $E$ is a Fréchet space and a $C^1$ function $\varphi : E \to \mathbb{R}$ is bounded below. Suppose $\mathcal{A}_k \neq \emptyset$ for some $k \geqq 1$. If $\varphi$ satisfies the Palais-Smale conditions at all levels $b = \mu_i, i = 1, \cdots k$, and $E^c$ is complete for each $c \in \mathbb{R}$, then $\varphi$ has at least $k$ distinct critical points.*

# Stabilizers of smooth functions on $2$-torus

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Let $M$ be a smooth compact surface. The group of diffeomorphisms $\mathcal{D}(M)$ naturally acts from the right on the space of smooth functions $C^\infty(M)$ by the following rule: $\gamma : C^\infty(M) \times \mathcal{D}(M) \to C^\infty(M)$, $\gamma(f,h) = f \circ h$. For the given smooth function $f \in C^\infty(M)$ we denote by $\mathcal{S}(f)$ and $\mathcal{O}(f)$ the stabilizer and the orbit of $f$ with respect to the action $\gamma$. Endow strong Whitney topologies on $C^\infty(M)$ and $\mathcal{D}(M)$; these topologies induce some topologies on $\mathcal{O}(f)$ and $\mathcal{S}(f)$. We denote by $\mathcal{D}_{\mathrm{id}}(M)$ and $\mathcal{O}_f(f)$ connected components of $\mathcal{D}(M)$ and $\mathcal{O}(f)$ which contain id and $f$ respectively; we also set $\mathcal{S}'(f) = \mathcal{S}(f) \cap \mathcal{D}_{\mathrm{id}}(M)$.

Let $\Gamma_f$ be a Kronrod-Reeb graph of a smooth function $f$. It is easy to see that each $h \in \mathcal{S}'(f)$ induces an automorphism $\rho(h)$ of the graph $\Gamma_f$, and the correspondence $\rho : \mathcal{S}'(f) \to \mathrm{Aut}(\Gamma_f)$ is a homeomorphism. The image of $\rho(\mathcal{S}'(f))$ in $\mathrm{Aut}(\Gamma_f)$ will be denoted by $G(f)$. More details can be found in [6].

In the series of papers [1]–[5] S. Maksymenko and the author described an algebraic structure of $\pi_1\mathcal{O}_f(f)$ for Morse functions on 2-torus. In my talk I am going to present algebraic structures of groups $\pi_0\mathcal{S}'(f)$ and $G(f)$ for Morse functions on 2-torus.

## References

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# Graphs and Riemann surfaces

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Riemann's moduli space $M_g$ is the space of isomorphism classes of genus g Riemann surfaces. It is a complex variety of dimension $3g-3$. I will discuss a connection between its rational homology groups $H_*(M_g)$ and the graph complexes introduced by Kontsevich in the 1990's, discovered in recent joint work with Chan and Payne (arXiv:1805.10186). In particular we show that the dimension of $H_{4g-6}(M_g)$ grows exponentially with $g$. It was known previously that $H_i(M_g) = 0$ for $i > 4g-6$.

# Entanglement and geometry of states of quantum many-particle systems

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In the talk we review the mathematical methods of the description of the evolution of states of quantum many-particle systems by means of the possible modifications of the density operator (density matrix). In particular, we consider some related problems of entanglement and the geometry of quantum states.

One of the approaches to describing the states of quantum systems of many particles consists in to describe states by means of a sequence of operators determined by the cluster expansions of density operators, which are interpreted as the correlation operators are governed by the hierarchy of the nonlinear evolution equations [1]. Such approach allows us to describe the evolution of correlations of systems in condensed states.

Moreover, we discuss an approach to the description of the evolution of states within framework of the state of a typical particle within a quantum system of many particles, i.e. the foundations of describing the evolution by the nonlinear kinetic equations are considered [2].

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# Braids, links, strings and algorithmic problems of topology

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V.V. Sharko in his book [5] has investigated functions on manifolds. Braids intimately connect with functions on manifolds. These connections are represented by mapping class groups of corresponding discs, by fundamental groups of corresponding punctured discs. and by some other topological or algebraic structures.

Below we follow to [1, 2, 3, 6] and references therein.

**Definition 1.** (Configuration spaces of the ordered sets of points). Let $M$ be a topological space and let $M^n$ be the product of $n$ spaces $M$ with the topology of the product. Put

$$\mathcal{F}_n(M) = \{(u_1, \ldots, u_n) \in M^n; u_i \neq u_j, i \neq j\}.$$

**Remark 2.** If $M$ is a topological space of the dimension $dim\ M$ (possibly with the boundary $\partial M$) then the dimension of $\mathcal{F}_n(M)$ is equal $n \cdot dim\ M$. The topological space $\mathcal{F}_n(M)$ is connected.

**Definition 3.** The fundamental group $\pi(\mathcal{F}_n(M))$ of the manifold $\mathcal{F}_n(M)$ is called the group of pure braids with $n$ strands.

Let now $M$ be a connected topological manifolds of the dimension $\geq 2$,

$$M^{in} = M \setminus \partial M, \qquad Q_m \subset M^{in},$$

$Q_m$ contains $m \geq 0$ points. Put

$$\mathcal{F}_{m,n}(M) = \mathcal{F}_n(M \setminus Q_m).$$

and for symmetric group $S_n$ put

$$\mathcal{G}_{m,n} = \mathcal{F}_{m,n}(M)/S_n.$$

**Definition 4.** The fundamental group $\pi(\mathcal{G}_{m,n})$ is called the braids group of the manifold $M \setminus Q_m$ with $n$ strands.

**Remark 5.** In the case of $M = \mathbb{R}^2$ we obtain groups of pure braids and braids in the sense of E. Artin and A. Markov.

Let $M$ be a three dimensional topological manifold possible with the boundary $\partial M$. Recall that a geometric link in $M$ is a locally flat closed one dimensional submanifold in $M$.

A.A. Markov [3] gave the description of the set of isotopic classes of oriented links in $\mathbb{R}^3$ in terms of braids. For manifolds of the dimension grater than 3 A.A. Markov [2] has proved the undecidability of the problem of homeomorphy.

For algorithmic and computer-algebraic investigations of braids, links and strings we have to represent corresponding data structures and algorithms. These data structures and algorithms are constructive mathematical objects in the sense of A. Markov. The processing of these constructive objects require corresponding constructive semantics.

A.A. Markov ( see [4] and references therein) began to construct the semantics. Follow to ([4] and references therein), and specialize Markov results to braids, links and strings we have

**Proposition 6.** *Let we have a description of braids, links and strings as constructive mathematical objects of the language $L_2$. Then any closed formula of the language $L_2$ which is inferred from the valid formula of the language $L_2$, is valid.*

We will present the interpretation of this Proposition on examples from [1, 3, 6, 7, 8].

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# Matrix manifolds as affine varieties


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Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, the skew $\mathbb{R}$-algebra of quaternions and write $M_{n,r}(\mathbb{K})$ (resp. $M_n(\mathbb{K})$) for the set of $n \times r$ (resp. $n \times n$) -matrices over $\mathbb{K}$. We aim to examine the structure of special matrix manifolds, Grassmann $G_{n,r}(\mathbb{K})$ and Stiefel manifolds $V_{n,r}(\mathbb{K})$, via their presentations as algebraic sets.

**Proposition 1.** *Let* $\mathbb{K} = \mathbb{R}, \mathbb{C}$ *or* $\mathbb{H}$*. Then:*
(1) $G_{n,r}(\mathbb{K}) = \{A \in M_n(\mathbb{K});\ A^2 = A,\ \bar{A}^t = A,\ \mathrm{tr}(A) = r\}$.
(2) $V_{n,r}(\mathbb{K}) = \{A \in M_{n,r}(\mathbb{K});\ \bar{A}^t A = I_r\}$.

This implies that $G_{n,r}(\mathbb{K})$ and $V_{n,r}(\mathbb{K})$ are $\mathbb{R}$-affine varieties. The main result is:

**Theorem 2.** (1) *The tangent bundle*

$$TG_{n,r}(\mathbb{K}) = \{(A, B) \in G_{n,r}(\mathbb{K}) \times M_n(\mathbb{K});\ \bar{B}^t = B,\ AB + BA = B\};$$

(2) *There is an algebraic isomorphism*

$$TG_{n,r}(\mathbb{K}) \approx \mathrm{Idem}_{r,n}(\mathbb{K}),$$

*where*

$$\mathrm{Idem}_{r,n}(\mathbb{K}) = \{A \in M_n(\mathbb{K});\ A^2 = A,\ \mathrm{rk}(A) = r\};$$

(3) *The* $\mathbb{C}$*-Zariski closure* $\overline{G_{n,r}(\mathbb{C})} = TG_{n,r}(\mathbb{C})$ *for* $G_{n,r}(\mathbb{C})$ *as an* $\mathbb{R}$*-affine variety.*

**Remark 3.** The Stiefel map $V_{n,r}(\mathbb{K}) \to G_{n,r}(\mathbb{K})$ given by $A \mapsto A\bar{A}^t$ for $A \in V_{n,r}(\mathbb{K})$ lead to a purely algebraic construction of the universal map

$$EU(\mathbb{K}) \longrightarrow BU(\mathbb{K}).$$

**Remark 4.** An extension of those results on some matrix manifolds over $\mathbb{K}$ being the Cayley algebra, and more generally, a composition algebra is planed as well.

# On the separability of the topology on the set of the formal power series

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Let we have the sequence $(a_n)_{n=0}^{\infty}$ of the elements from any ring $K$. The expression $a_0 + a_1x + a_2x^2 + ... = \sum_{i=0}^{\infty} a_i x^i$, where $+$ and $x$ are the formal symbols, $a_i$ are coefficients, $a_0$ – intercept term is called the formal power series.

Sometimes the expression $f(x) = a_0 + a_1x + a_2x^2 + ...$ is called the another form of the representation of the sequence $(a_n)_{n=0}^{\infty}$. It is convenient to consider the finite sequence as the infinite one. The corresponding formal power series is called the polynomial. The set of the formal power series with coefficients from $K$ is denote by $K[[x]]$.

The set $K[[x]]$ is a ring with identity with respect to the addition and the multiplication of the formal power series:

$$\sum_{i=0}^{\infty} a_i x^i + \sum_{i=0}^{\infty} b_i x^i = \sum_{i=0}^{\infty} (a_i + b_i) x^i$$

and

$$(\sum_{i=0}^{\infty} a_i x^i)(\sum_{i=0}^{\infty} b_i x^i) = \sum_{k=0}^{\infty}(\sum_{i+j=k} a_i b_j)x^k.$$

The sets $I_k = x^k K[[x]], k = 0, 1, 2, ...$ are the principal ideals of the ring $K[[x]]$. If these ideals are considered as the open sets, then they and the empty set form the topological structure on the $K[[x]]$.

In the topological space $(R[[x]], \tau)$ if the formal power series $f(x)$ has nonzero intercept term, then its neighborhood is $R[[x]]$. If $f(x)$ has no intercept term, then one can consider any $I_k$ as the neighborhood, where $k$ is not greater than the power of the minimal monomial term of the series $f(x)$.

**Proposition 1.** *The topological space* $(R[[x]], \tau)$ *is separable.*

In this space the set $M$ of all polynomials with the integer coefficients is denumerable as it is equal to $\bigcup_{n\in N} P_n$, where $P_n$ is the set of all polynomials with the integer coefficients the powers of which are less than or equal to $n$ which is equivalent to Cartesian product $Z \times Z \times ... \times Z$, where $Z$ is repeated $n+1$ times. Moreover the closure $\bar{M}$ is equal to $R[x]$.

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# On the monoid of cofinite partial isometries of a finite power of positive integers with the usual metric

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We follow the terminology of [1, 2]. For any positive integer $n$ by $\mathcal{S}_n$ we denote the group of permutations of the set $\{1, \ldots, n\}$.

A partial transformation $\alpha\colon (X, d) \rightharpoonup (X, d)$ of a metric space $(X, d)$ is called *isometric* or a *partial isometry*, if $d(x\alpha, y\alpha) = d(x, y)$ for all $x, y \in \operatorname{dom}\alpha$.

For an arbitrary positive integer $n \geqslant 2$ by $\mathbb{N}^n$ we denote the $n$-th power of the set of positive inters $\mathbb{N}$ with the usual metric:

$$d((x_1, \cdots, x_n), (y_1, \ldots, y_2)) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$

Let $\mathbf{I}\mathbb{N}^n_\infty$ be the set of all partial cofinite isometries of $\mathbb{N}^n$. It is obvious that $\mathbf{I}\mathbb{N}^n_\infty$ with the operation of composition of partial isometries is an inverse submonoid of the symmetric inverse monoid $\mathcal{I}_\mathbb{N}$ over $\mathbb{N}$ and later by $\mathbf{I}\mathbb{N}^n_\infty$ we shall denote the *monoid of all partial cofinite isometries of* $\mathbb{N}^n$.

**Theorem 1.** *For any positive integer $n \geqslant 2$ the group of units $H(\mathbb{I})$ of the monoid $\mathbf{I}\mathbb{N}_{\infty}^{n}$ is isomorphic to the symmetric group $\mathcal{S}_n$. Moreover, every element of $H(\mathbb{I})$ is induced by a permutation of the set $\{1, \ldots, n\}$.*

**Lemma 2.** *Let $n$ be any positive integer $\geqslant 2$. Let $\alpha$ be an arbitrary element of the monoid $\mathbf{I}\mathbb{N}_{\infty}^{n}$. Then there exists the unique element $\sigma_\alpha$ of the group of units $H(\mathbb{I})$ and the unique idempotents $\varepsilon_{l(\alpha)}$ and $\varepsilon_{r(\alpha)}$ of the semigroup $\mathbf{I}\mathbb{N}_{\infty}^{n}$ such that $\alpha = \sigma_\alpha \varepsilon_{l(\alpha)} = \varepsilon_{r(\alpha)} \sigma_\alpha$.*

If $S$ is an inverse semigroup then the semigroup operation on $S$ determines the following partial order $\preccurlyeq$ on $S$: $s \preccurlyeq t$ if and only if there exists $e \in E(S)$ such that $s = te$. This order is called the *natural partial order* on $S$ [4].

**Theorem 3.** *Let $n$ be any positive integer $\geqslant 2$. Let $\alpha$ and $\beta$ be elements of the semigroup $\mathbf{I}\mathbb{N}_{\infty}^{n}$. Let $\alpha = \sigma_\alpha \varepsilon_{l(\alpha)} = \varepsilon_{r(\alpha)} \sigma_\alpha$ and $\beta = \sigma_\beta \varepsilon_{l(\beta)} = \varepsilon_{r(\beta)} \sigma_\beta$ for some elements $\sigma_\alpha$ and $\sigma_\beta$ of the group of units $H(\mathbb{I})$ and idempotents $\varepsilon_{l(\alpha)}$, $\varepsilon_{r(\alpha)}$, $\varepsilon_{l(\beta)}$ and $\varepsilon_{r(\beta)}$ of the semigroup $\mathbf{I}\mathbb{N}_{\infty}^{n}$. Then $\alpha \preccurlyeq \beta$ in $\mathbf{I}\mathbb{N}_{\infty}^{n}$ if and only if $\sigma_\alpha = \sigma_\beta$, $\varepsilon_{l(\alpha)} \preccurlyeq \varepsilon_{l(\beta)}$ and $\varepsilon_{r(\alpha)} \preccurlyeq \varepsilon_{r(\beta)}$ in $E(\mathbf{I}\mathbb{N}_{\infty}^{n})$.*

A congruence $\mathfrak{C}$ on a semigroup $S$ is called a *group congruence* if the quotient semigroup $S/\mathfrak{C}$ is a group. If $\mathfrak{C}$ is a congruence on a semigroup $S$ then by $\mathfrak{C}^\sharp$ we denote the natural homomorphism from $S$ onto the quotient semigroup $S/\mathfrak{C}$. Every inverse semigroup $S$ admits a *least* (*minimum*) *group congruence* $\mathfrak{C}_{\mathbf{mg}}$:

$$a \mathfrak{C}_{\mathbf{mg}} b \text{ if and only if there exists } e \in E(S) \text{ such that } ae = be$$

(see [3, Lemma III.5.2]).

**Theorem 4.** *Let $n$ be any positive integer $\geqslant 2$. Then the quotient semigroup $\mathbf{I}\mathbb{N}_{\infty}^{n}/\mathfrak{C}_{mg}$ is isomorphic to the group $\mathcal{S}_n$ and the natural homomorphism $\mathfrak{C}_{mg}^\sharp \colon \mathbf{I}\mathbb{N}_{\infty}^{n} \to \mathbf{I}\mathbb{N}_{\infty}^{n}/\mathfrak{C}_{mg}$ is defined in the following way: $\alpha \mapsto \sigma_\alpha$.*

The following theorem gives the description of Green's relations $\mathcal{R}$, $\mathcal{L}$, $\mathcal{H}$, $\mathcal{D}$ and $\mathcal{J}$ on $\mathbf{I}\mathbb{N}_{\infty}^{n}$.

**Theorem 5.** *Let $n$ be any positive integer $\geqslant 2$ and $\alpha, \beta \in \mathbf{I}\mathbb{N}_{\infty}^{n}$. Then the following statements hold:*

(*i*) *$\alpha \mathcal{L} \beta$ if and only if there exists an element $\sigma$ of the group of units $H(\mathbb{I})$ of $\mathbf{I}\mathbb{N}_{\infty}^{n}$ such that $\alpha = \sigma\beta$;*
(*ii*) *$\alpha \mathcal{R} \beta$ if and only if there exists an element $\sigma$ of the group of units $H(\mathbb{I})$ of $\mathbf{I}\mathbb{N}_{\infty}^{n}$ such that $\alpha = \beta\sigma$;*
(*iii*) *$\alpha \mathcal{H} \beta$ if and only if there exist elements $\sigma_1$ and $\sigma_2$ of the group of units $H(\mathbb{I})$ of $\mathbf{I}\mathbb{N}_{\infty}^{n}$ such that $\alpha = \sigma_1\beta$ and $\alpha = \beta\sigma_2$;*
(*iv*) *$\alpha \mathcal{D} \beta$ if and only if there exist elements $\sigma_1$ and $\sigma_2$ of the group of units $H(\mathbb{I})$ of $\mathbf{I}\mathbb{N}_{\infty}^{n}$ such that $\alpha = \sigma_1\beta\sigma_2$;*
(*v*) *$\mathcal{D} = \mathcal{J}$ on $\mathbf{I}\mathbb{N}_{\infty}^{n}$;*
(*vi*) *every $\mathcal{J}$-class in $\mathbf{I}\mathbb{N}_{\infty}^{n}$ is finite and consists of incomparable elements with the respect to the natural partial order $\preccurlyeq$ on $\mathbf{I}\mathbb{N}_{\infty}^{n}$.*

**Corollary 6.** *$\mathbf{I}\mathbb{N}_{\infty}^{n}$ is an $E$-unitary $F$-inverse semigroup for any positive integer $n \geqslant 2$.*

The following theorem describes the structure of the semigroup $\mathbf{I}\mathbb{N}_\infty^n$.

**Theorem 7.** *Let $n$ be any positive integer $\geqslant 2$. Then the semigroup $\mathbf{I}\mathbb{N}_\infty^n$ is isomorphic to the semidirect product $\mathcal{S}_n \ltimes_{\mathfrak{h}} (\mathcal{P}_\infty(\mathbb{N}^n), \cup)$ of free semilattice with the unit $(\mathcal{P}_\infty(\mathbb{N}^n), \cup)$ by the symmetric group $\mathcal{S}_n$.*

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# Properties of some algebras of entire functions of bounded type, generated by a countable set of polynomials on a Banach space


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Let $X$ be a complex Banach space. Let $\mathbb{P} = \{P_1, P_2, \ldots, P_n, \ldots\}$ be a countable set of algebraically independent continuous $n-$homogeneous complex-valued polynomials on $X$ for every positive integer $n$. Let us denote by $H_{b\mathbb{P}}(X)$ the closed subalgebra, generated by the elements of $\mathbb{P}$, of the Fréchet algebra $H_b(X)$ of all entire functions of bounded type on $X$.

In this talk we will discuss some properties of the algebra $H_{b\mathbb{P}}(X)$. For instance, we will show that every $f \in H_{b\mathbb{P}}(X)$ can be uniquely represented in the following form

$$f(x) = f(0) + \sum_{n=1}^{\infty} \sum_{k_1+2k_2+\ldots+nk_n=n} a_{k_1,k_2,\ldots,k_n} (P_1(x))^{k_1} (P_2(x))^{k_2} \cdots (P_n(x))^{k_n},$$

where $x \in X$, $a_{k_1,k_2,\ldots,k_n} \in \mathbb{C}$ and $k_1, k_2 \ldots, k_n \in \mathbb{N} \bigcup \{0\}$. Consequently, the spectrum (the set of all continuous linear multiplicative functionals) of the algebra $H_{b\mathbb{P}}(X)$ is in one-to-one correspondence with some set of sequences of complex numbers. We will prove the upper estimate for sequences of this set. We will also describe the spectrum of algebra $H_{b\mathbb{P}}(X)$ in case when $X$ is a closed subspace of the space $l_\infty$ such that $X$ contains the space $c_{00}$ for some special form of the set $\mathbb{P}$.

# Optimal Morse Flows on non-orientable 3-manifolds

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A flow is called a *Morse flow* if the following conditions hold true:

1) it has finitely many singular points which are non-degenerate;

2) stable and unstable manifolds with singular points have a transversal intersection;

3) $\alpha-$ and $\omega-$limit sets of each trajectory are singular points.

Let $M$ be a smooth closed non-orientable 3-manifold. *A Morse flow diagram* takes the form of a surface and two sets of circles embedded in it. The surface $F$ is the boundary of the regular neighborhood of sources and stable manifolds with singular points of index 1.

On the surface $F$, we have the following sets of circles:

1) circles $u$, which are intersections of unstable manifolds of singular points of index 1 with $F$;

2) circles $v$, which are intersections of stable manifolds of singular points of index 2 with $F$.

*Morse flow diagram* on a three-dimensional manifold is the 3-tuple $(F, u, v)$ consisting of a surface, a set of circles. Two Morse flow diagrams are said to be *equivalent* if there is a surface homeomorphism that maps the sets of arcs and circles into sets of arcs and circles of the same type and preserves framing. According to A.Prishlyak, two Morse flows on 3-manifolds are topologically equivalent if and only if their diagrams are equivalent.

A Morse flow is called *optimal* if it has the minimum number of singular points and trajectories between the saddles. An optimal (polar) Morse flow diagram on a closed 3-manifold is similar to a Heegaard diagram.

We show that if the Heegaard complexity of a non-orientable 3-manifold $M$ is no more than 5, then $M$ is homeomorphic to $L_{p,q}\#S^1\widetilde{\times}S^2$. There exists a Heegaard diagram of complexity 6 of a closed non-orientable 3-manifold which is not homeomorphic to $L_{p,q}\#S^1\widetilde{\times}S^2$.

We show that up to a topological equivalence, there exist

- a unique optimal Morse flow on $S^1\widetilde{\times}S^2$,
- 2 flows on $L_{2,1}\#S^1\widetilde{\times}S^2$,
- 2 flows on $L_{3,1}\#S^1\widetilde{\times}S^2$,
- 3 flows on $L_{4,1}\#S^1\widetilde{\times}S^2$,
- 3 flows on $L_{5,1}\#S^1\widetilde{\times}S^2$ and
- 3 flows on $L_{5,2}\#S^1\widetilde{\times}S^2$.

There are no other optimal flows on closed non-orientable 3-manifolds of complexity less than 6.

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# Functions with isolated critical points on the boundary of non-oriented surface


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Let $f$ be simple smooth function with isolated critical points on the boundary of smooth compact connected non-oriented surface $M$ which are also isolated critical points of restriction of function $f_\partial M$ to the boundary $\partial M$.

Let us consider the neighborhood of a critical point $p_0$ bounded by $f^{-1}(-\varepsilon)$, $f^{-1}(\varepsilon)$ for some small enough $\varepsilon > 0$, by some trajectories of a gradient field and by the boundary $\partial M$. The parts of the surface where $f > 0$ and $f < 0$ will be called the positive and negative sectors of function $f$. We depict these sectors by shaded and unshaded ones. The obtained surface has the structure of $(2k+2)$–gon. If we extend this neighborhood to the neighborhood of a critical level, we get the neighborhood which is homeomorphic to a polygon with glued sides by linear homeomorphism.

Thus, atom has the structure of $(2k+2)$–gon (see [4] Figures 5-1, 5-2).

We put a circle with matched points corresponding to the previously described polygon. This circle is the boundary $(2k+2)$–gon and matched points are the points on the circle belonging to the intersection of shaded and unshaded sectors (in other words, matched points belong to the critical level). We connect two matched points by a chord if and only if correspondent sides of polygon become glued after continuations of critical level neighborhood. In what follows we get the circle with a matched points. Further fix the orientation on the boundary to numerate the matched points on the circle. If we change the orientation, we get the equivalent atom. Then we numerate matched points in the following way: a point corresponding to a critical one $p_0$ we denote by $Q_0$, and the rest of points we numerate according to the orientation of the boundary beginning with $Q_1$ up to $Q_k$ and point $Q_0$ we consider as the point of reference. These points divide the circle into $k+1$ black (thick) and grey (thin) arcs. These arcs correspond to positive and negative sections.

Then, every atom can be defined by the circle with $k+1$ matched points and $l$ chords (for some $l \in \{0, 1, 2, \dots, [\frac{k}{2}]\}$). Also one matched point corresponds to a critical point.

**Definition 1.** *A chord diagram of a saddle critical level* of the function defined on a smooth compact surface with the boundary is the circle with the following elements:

(1) matched points, which are enumerated;
(2) chords, the ends of which are different matched points except for $Q_0$;
(3) coloration of arcs such that each two neighbor arcs with the exception of arcs near point $Q_0$, are of different colors.

Note that chord diagram defines f–atom and if we consider only elements (1) and (2) then chord diagram defines atom.

**Definition 2.** Two chord diagrams are called *equivalent* if they can be obtained one from another by turn or symmetry preserving the elements (1) — (3).

**Definition 3.** *A free matched point* on chord diagram is the one which is not connected with another matched points by chord.

Chord diagrams are also considered in papers [1], [2], [3].

Further we consider a smooth surface $M$ with a component of the boundary $\partial M$ and a simple smooth function $f \in \Omega(M)$.

**Definition 4.** We call a function $f \in \Omega(M)$ *optimal on the surface* $M$ if it has the minimum possible number of critical points on $M$ among all functions from $\Omega(M)$.

**Definition 5.** A homeomorphism

$$h : [0, k] \to S^1 \bigcup Int\{l_{mn} | m, n \in \{1, \dots, k\}\}$$

will be called a *full way* between free points $Q_0$ and $Q_{i^*}$ (for some $i \in \overline{1, k}$) if it satisfies the conditions:

1) $h(0) = Q_0$, $h(k) = Q_{i^*}$;
2) $\forall t \in \{1, 2, \dots, k-1\} : h(t) \in \{Q_1, Q_2, \dots, Q_{i^*-1}, Q_{i^*+1}, \dots, Q_k\}$;
3) $\forall j \in \{1, 2, \dots, k-1\}$ $\forall t \in (j, j+1)$: $f(t)$ belongs either to the interior of arc, or to the interior of chord;
4) the direction can not be changed during the moving on fixed chord diagram.

**Theorem 6.** *A chord diagram of saddle critical level of optimal function on non-oriented surface with one component of the boundary satisfies the following conditions:*

1) *there exist at least one chord which divides the circle into two arcs, each of which contains the odd number of matched points;*
2) *the chord diagram has $k+1 = 2n+2$ matched points (for some integer $n$, $n \geq 1$) and there exist exactly two free points, one of which is $Q_0$;*
3) *there exist exactly two full ways between free points.*

**Proposition 7.** *Optimal functions are topologically equivalent if and only if their chord diagrams of saddle critical levels are equivalent.*

**Proposition 8.** *If a chord diagram satisfies the conditions* 1)–3) *of Theorem 6, then there exists an optimal function, chord diagram of saddle critical level of which coincides with the first one.*

**Theorem 9.** *It was defined the number of topological non-equivalent optimal functions on non-oriented surface with one component of the boundary in cases:*

- *1 on surface by genus 1;*
- *3 on surface by genus 2;*
- *20 on surface by genus 3.*

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# Enumeration of topologically non-equivalent functions with one degenerate saddle critical point on two-dimensional torus


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Let $(N, \partial N)$ be a smooth surface with possible empty boundary whihc is divided into two open-closed subsets $\partial_- N$ and $\partial_+ N$, so $\partial N = \partial_- N \sqcup \partial_+ N$. Let $C^\infty(N)$ denote the space of infinitely differentiable functions on $N$, all critical points of which are isolated and lie in the interior of $N$, and, moreover, on the connected components of $\partial_- N$ ($\partial_+ N$) the functions from $C^\infty(N)$ take the same values $a$ ($b$ accordingly).

Two functions $f_1$ and $f_2$ from the space $C^\infty(N)$ are called topologically equivalent if there are homeomorphisms $h : N \to N$ and $h' : R^1 \to R^1$ ($h'$ preserves orientation) such that $f_2 = h' \circ f_1 \circ h^{-1}$. If $h$ preserves of the orientation, the functions $f_1$ and $f_2$ are called topologically conjugated (eg. [4]) or $O$-topologically equivalent (eg. [6]).

It is known [4] that a function $f \in C^\infty(N)$, in a certain neighborhood of its isolated critical point $x \in N$ (which is not a local extremum) for which the topological type of level lines changes in passing through $x$, is reduced by a continuous change of coordinates to the form $f = \mathrm{Re} z^n + c$, $n \geq 2$ (are called «essentially» critical point) or $f = \mathrm{Re} z$ if the topological type of level lines does not change in passing through $x$. The number of essentially critical points $x_i$ of the function $f$, together with the values of $n_i$ (exponents in there presentation $f = \mathrm{Re} z^{n_i} + c_i$ in the neighborhoods of the critical points $x_i$), are called the topological singular type of the function $f$.

The problem of the topological equivalence of functions from the class $C^\infty(N)$ with the fixed number of critical points was completely solved by V.V. Sharko in [5] and it was established that a finite number of topologically nonequivalent functions of this class exist. However, it should be noted that the task about calculation of topologically non-equivalent functions with the fixed topological singular type is rather complicated and is still **unsolved**.

When considering functions from the class $C(M_g) \subset C^\infty(N)$ that possess **only one** essentially critical point $x_0$ (degenerate critical point of the saddle type) in addition to local maxima and minima on oriented surface $M_g$ of genus $g \geq 0$, then the problem of counting the number of such non-equivalent functions is greatly simplified. It is well known [4] that $\forall f \in C(M_g)$ the Poincare index of a critical point $x_0$, is equal $\mathrm{ind}^f(x_0) = 1 - n$, where $n = 2g - 1 + \lambda$ and $\lambda$ is a total number of local maxima and minima.

Let $C_n(M_g)$ be a class of functions from $C(M_g)$ which, in addition to local maxima and minima, have only one essentially critical point, whose the Poincare index is equal $(1 - n)$. Denote the class of functions from $C(M_g)$ that possess one essentially critical point, $k$ local maxima and $l$ local minima by $C_{k,l}(M_g)$. Then it is clear that $\forall f \in C_{k,l}(M_g)$ $n = 2g - 1 + k + l$.

In the general case, for natural $g, k, l$ (or $k$, $l$, and $n = 2g + k + l - 1$), the problem of calculating the number of topologically non-equivalent functions from the class $C_{k,l}(M_g)$ also has proved to be quite a difficult and unsolved problem to date.

The task of calculating the number of topologically non-equivalent functions from the class $C_{1,1}(M_g)$ (for genus $g \geq 1$) was completely solved in [6].

The exact formulas established in [2] solve completely the tasks of calculating the number of $O$-topologically non-equivalent and the number of topologically non-equivalent functions from the class $C_n(M_0)$.

In [7], for natural $k$ and $l$ solved completely the problems of calculating the numbers $O$-topologically and topologically non-equivalent functions from the class $C_{k,l}(M_0)$ (on two-dimensional sphere).

For two-dimensional torus $T^2$ the problems of the enumeration of $O$-topologically and of topologically non-equivalent functions are solved only for class $C_{1,l}(T^2)$, $C_{2,l}(T^2)$ and $C_{3,l}(T^2)$ in [8, 9, 10] accordingly. In general case, for fixed natural $k$ and $l$, the task is also still unsolved.

By using the results of [1] and [3], we can establish the validity of the following statement:

**Theorem 1.** *For the natural $n \geq 3$ the number of $O$-topologically non-equivalent functions from the class $C_n(T^2)$ can be calculated by the relations*

$$t^*(n) = \frac{1}{n}\left(t(n) + a(n) + 2b\left(\tfrac{2n}{3}\right) + 2c\left(\tfrac{n}{2}\right) + 2d\left(\tfrac{n}{3}\right)\right), \tag{1}$$

*where*

$$t(n) = \frac{1}{6} C_{n-1}^2 C_{2(n-1)}^{n-1}, \qquad u(p) = \frac{(2p)!}{p!p!} = C_{2p}^p;$$

$$a(2p+1) = 0, \qquad a(2p) = \tfrac{1}{6} p(p-1) \cdot C_{2p}^p = \tfrac{1}{6} p(p-1) \cdot u(p);$$

$$b(2p+1)=0, \qquad b(2p)=(2p-1)\cdot C_{2(p-1)}^{p-1}=(2p-1)\cdot u(p-1);$$

$$c(2p+1)=d(2p+1)=0, \qquad c(2p)=d(2p)=p\cdot C_{2p}^{p}=p\cdot u(p).$$

By using the results of [8, 9, 10], we have the following values

| | n | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| $(k;l)$ $_{k,l\in N,\ k+l=n-1}$ | $(1;n-2)$ | 1 | 2 | 3 | 7 | 10 | 17 | 24 | 34 |
| | $(2;n-3)$ | — | 2 | 8 | 31 | 80 | 187 | 374 | 698 |
| | $(3;n-4)$ | — | — | 3 | 31 | 150 | 557 | 1634 | 4172 |
| | $(4;n-5)$ | — | — | — | 7 | 80 | 557 | 2616 | 9724 |
| | $(5;n-6)$ | — | — | — | — | 10 | 187 | 1634 | 9724 |
| | $(6;n-7)$ | — | — | — | — | — | 17 | 374 | 4172 |
| | $(7;n-8)$ | — | — | — | — | — | — | 24 | 698 |
| | $(8;n-9)$ | — | — | — | — | — | — | — | 34 |
| $t^*(n)=\sum\limits_{k+l=n-1} t^*_{k,l}$ | | 1 | 4 | 14 | 76 | 330 | 1522 | 6680 | 29256 |

Table 1.1. Number $t^*_{k,l}$ of $O$-topologically non-equivalent functions from the class $C_{k,l}(T^2)$

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# Weak $R$-spaces and uniform limit of sequences of discontinuous functions

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We will discuss a new notion of a weak R-space, consider some examples and study relations between R-spaces, weak R-spaces and uniform limits of sequences of maps from different functional classes (Baire one, homotopic Baire one, Borel, Darboux, etc.)

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# Exact Morse functions on Kendall shape spaces

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We introduce and investigate several Morse functions on Kendall shape spaces. The main attention is given to a properly normalized oriented area and Coulomb potential on shape spaces of planar $n$-gons. It is easy to verify that convex and equiangular shapes of regular $n$-gons are critical points of the foregoing functions. We prove that, in many cases, these critical points are non-degenerate, and compute their Morse indices.

It follows that, for odd $n$, the normalized oriented area is an exact Morse function on the Kendall shape space of planar $n$-gons. For even $n$, we are only able to show that the normalized oriented area has the minimal possible number of critical points equal to the Lusternik-Schnirelmann category of the shape space considered. The proofs use our earlier results on Morse functions on moduli spaces of planar linkages obtained jointly with G.Panina and D. Siersma.

For $n = 4, 5$, we also show that any two points in the shape space can be connected by a differentiable curve consisting of equilibrium configurations of the normalized Coulomb potential with varying charges of vertices.

Possible analogs and generalizations to spatial shape spaces will also be mentioned.

# Diffeomorphisms groups of certain singular foliations on lens spaces

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Let

$$T = D^2 \times S^1 = \{(x, y, w) \in \mathbb{R}^2 \times \mathbb{C} \mid x^2 + y^2 \leq 1,\ |w| = 1\}$$

be a solid torus, $C_r = \{z \in D^2 \mid |z| = r\} \subset D^2$, $r \in [0, 1]$ and

$$\mathcal{F}_T = \{C_r \times S^1\}_{r \in [0,1]}$$

be a foliation on $T$ into 2-tori parallel to the boundary and one singular circle $C_0 \times S^1$, which is the central circle of the torus $T$.

Denote by $\mathcal{D}(\mathcal{F}_T, \partial T)$ the group of leaf preserving diffeomorphisms of $T$, which are fixed on $\partial T$.

**Theorem 1.** *The group $\mathcal{D}(\mathcal{F}_T, \partial T)$ is contractible.*

Let $L_{p,q}$ be a lens space, that is, a three-manifold obtained by gluing two solid tori $T_1 = T_2 = D^2 \times S^1$ via some diffeomorphism $\phi : \partial T_1 \to \partial T_2$ of their boundaries.

The topological type of the lens space is determined by the isotopy class of the image of the meridian of $\partial T_1$ in $\partial T_2$.

Notice that each of the solid tori $T_i$, $i = 1, 2$, has a foliation $\mathcal{F}_i$ on two-dimensional tori

$$\mathcal{F}_{T_i} = \{C_r \times S^1\}_{r \in [0,1]}$$

with one singular leaf $C_0 \times S^1$ being the central circle of the torus $T_i$.

Since $\partial T_i$ is a leaf of foliation $\mathcal{F}_{T i}$, we see that the foliations on $\mathcal{F}_{T_1}$ and $\mathcal{F}_{T_2}$ determine a certain foliation $\mathcal{F}_{p,q} = \mathcal{F}_1 \cup \mathcal{F}_2$ on $L_{p,q}$ into 2-tori parallel to the boundary and two singular circles being the central circles $C_1$ and $C_2$ of the solid tori $T_1$ and $T_2$.

Denote by $\mathcal{D}(\mathcal{F}_{p,q})$ the group of leaf preserving diffeomorphisms of $L_{p,q}$ which also preserve their orientations .

**Theorem 2.** *If $L_{p,q} = S^2 \times S^1$, then the group $\mathcal{D}(\mathcal{F}_{p,q})$ is homotopy equivalent to $\mathbb{Z} \times S^1 \times S^1$. In all other cases, $\mathcal{D}(\mathcal{F}_{p,q})$ is homotopy equivalent to $S^1 \times S^1$.*

# Dynamical system of inverse heat conduction via Direct method of Lie-algebraic discrete approximations

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The dynamical system describing inverse heat conduction has applications in the different fields: image processing, signal processing, eliminating of diffusion. Hence effective numerical solution is an important problem besides the variety of different approaches.

Some of the methods for numerical study of dynamical systems [1] can provide factorial convergence and relatively high accuracy of approximated solution [2, 3]. For instance, the Generalized method of Lie-algebraic discrete approximations (GMLADA) provides factorial convergence rate for all variables: for space and for the time variable as well [4].

According to [5], computational properties of [4] can be enhanced via Direct method of Lie-algebraic discrete approximations (DMLADA), so we can construct the numerical scheme for solving heat equation having the same accuracy with significantly less arithmetic operations.

Considering a bounded domain $\Omega := (a, b) \in R$, time limit $T < +\infty$, cylinder $Q_T = \Omega \times (0, T]$ we take the Banach space $V = W^{\infty,\infty}\left(\overline{Q_T}\right)$ and formulate the Cauchy problem for dynamical system

$$\begin{cases} \text{find function } u = u(x,t) \in V \text{ such, that:} \\ \dfrac{\partial u}{\partial t} = -a\dfrac{\partial^2 u}{\partial x^2}, \ \forall (x,t) \in Q_T, \\ u|_{t=0} = \varphi, \varphi \in W^{\infty,\infty}\left(Q_T\right), \end{cases} \tag{1}$$

where the constant $a \in R, a > 0$ denotes the heat conduction coefficient and $\varphi = \varphi(x)$ denotes the initial condition, and space $W^{\infty,\infty}\left(\overline{Q_T}\right)$ denotes the functional space in which all functions and its derivatives up to arbitrary order are bounded in the domain $\overline{Q_T}$, i.e.:

$$W^{\infty,\infty}\left(\overline{Q_T}\right) = \{u : Q_T \to \mathbb{R} : D^\alpha u \in L^\infty(Q_T), \forall \alpha \in \mathbb{N}\}.$$

In general, current problem is ill posed.

The main prerequisite of the Lie-algebraic method is that differential operator of the equation should be the element of universe enveloping Heisenberg-Weyl's algebra with basis elements from the Lie algebra $\{1, x, d/dx\}$, i.e. differential operator for the problem must be superposition and/or linear combination of these base elements of Lie algebra. As a next step we there are introduced the finite dimensional discrete quasi representations of $G = \{1, x, d/dx\}$ as matrices $G_h = \{I, X, Z\}$ which act in the finite dimensional space.

The idea of DMLADA consists in the use of analytical approaches [5], in particular the method of a small parameter, to construct an approximate analytic solution of a problem (1) in the form of a power series in a time variable:

$$
\begin{aligned}
u_{n/2}(x,t) &= \sum_{k=0}^{n/2}\left(\tilde{u}_k \frac{t^k}{k!}\right) \\
&= \varphi - a\varphi'' t + a^2\varphi^{(4)}\frac{t^2}{2!} + \cdots + (-1)^{n/2}a^{n/2}\varphi^{(n)}\frac{t^{n/2}}{(n/2)!}.
\end{aligned} \tag{2}
$$

The corresponding discrete series was constructed for (2) using the finite dimensional quasi-representations $G_h = \{1, Z, X\}$ of elements of the Lie algebra $G = \{1, \partial/\partial x, x\}$:

$$
\begin{aligned}
u_{n/2,h}(t) &= \sum_{k=0}^{n/2}\left(\tilde{u}_{k,h} \frac{t^k}{k!}\right) \\
&= \varphi_h - aZ^2\varphi_h t + a^2 Z^4\varphi_h\frac{t^2}{2!} + \cdots + (-1)^{n/2}a^{n/2}Z^n\varphi_h\frac{t^{n/2}}{(n/2)!},
\end{aligned} \tag{3}
$$

where the matrix $Z$ approximates the differential operator $d/dx$. The series (3) is finite, since the matrix $Z$ is nilpotent [2].

Let us consider the cylinder norm for the function $v = v(x) : \mathbb{R} \to \mathbb{R}$: as a following functional:

$$
\|v\|_{V_h}^2 = \frac{1}{n+1}\sum_{i=0}^{n} v^2(x_i),
$$

being a norm in the finite dimensional space $V_h$. One can verify that the following inequality holds [5]

$$
\|v\|_{V_h} \le \|v\|_\infty.
$$

**Theorem 1** (Convergence of the direct Lie-algebraic numerical scheme)**.** *Suppose* $u = u(x,t)$ *is the solution of the problem* (1)*, and let*

$$
u_n = \sum_{k=0}^{n/2}\left((-1)^k a^k \varphi^{(2k)}\frac{t^k}{k!}\right)
$$

*be the Taylor expansion of the solution and*

$$
u_h = \sum_{j=0}^{n}\left[\left(\sum_{k=0}^{n/2}\left((-1)^k a^k Z^{2k}\varphi_h\frac{t^k}{k!}\right)\right) l_j(x)\right]
$$

*be the finite dimensional solution. Then built numerical scheme* (3) *is convergent having the factorial rate of convergence:*

$$
\|u - u_h\|_{B_h} \le \frac{T^{n/2+1}}{\left(\frac{n}{2}+1\right)!}\left\|\frac{\partial^{n+1}u}{\partial t^{n+1}}\right\|_\infty + \frac{(2\max\{a, \operatorname{diam}\Omega, T\})^{n+1}}{4\,(n/2-1)!}\left\|\varphi^{(n+1)}\right\|_\infty.
$$

Computational experiments have shown that with the same accuracy and convergence indicators that are characteristic for the generalized method of Lie-algebraic discrete approximations, we succeeded in significantly reducing the number of arithmetic operations using our approach.

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# Projective invariants of rational mappings


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We consider the $k$-jet spaces $\mathbf{J}^k$ as $k$-jets of analytical mappings $\mathbb{C}\boldsymbol{P}^1 \to \mathbb{C}\boldsymbol{P}^1$ equipped with adjoint $\mathbb{P}\mathbf{SL}_2(\mathbb{C})$-action:

$$f \mapsto A \circ f \circ A^{-1},$$

where $A \in \mathbb{P}\mathbf{SL}_2(\mathbb{C})$ (cf. [3]).

The representations of corresponding Lie algebras by vector fields on $\mathbf{J}^0$ is the following:

$$\langle \partial_z + \partial_u, z\partial_z + u\partial_u, z^2\partial_z + u^2\partial_u \rangle.$$

**Theorem 1.** (1) *The field of adjoint invariants* [1], [2], *is generated by differential invariants of the second and third orders*

$$J_2 = u_1^{-3}\left(\left(z-u\right)u_2 + 2u_1\left(u_1+1\right)\right)^2,$$

$$J_3 = \left(z-u\right)^2 u_1^{-2}u_3 + 6\left(z-u\right)u_1^{-2}\left(u_1+1\right)u_2 + 6\left(u_1 + u_1^{-1}\right),$$

*and invariant derivation*

$$\nabla = \frac{\left(z-u\right)u_1u_2 + 2u_1^2\left(u_1+1\right)}{\left(z-u\right)\left(2u_1u_3 - 3u_2^2\right)}\frac{d}{dz}.$$

(2) *This field separates regular orbits.*

Equation $J_2\left(f\right) = 0$ has solutions of the form:

$$f\left(z\right) = \frac{ax+b}{cx+d}$$

where matrix

$$A = \begin{Vmatrix} a & b \\ c & d \end{Vmatrix}$$

has zero trace.

In general, values of invariant $J_2$ on linear fractional maps is equal to

$$J_2\left(f\right) = 4\frac{\left(trA\right)^2}{\det A},$$

and

$$J_3\left(f\right) = 6\frac{trA^2}{\det A} + 24.$$

We say that a rational mapping $f\left(z\right)$ is *singular* if $f\left(z\right)$ is linear fractional.

There exists an irreducible polynomial (we call it *generating polynomial*)

$$P_f\left(X, Y\right) \in \mathbb{C}[X, Y]$$

such that $P_f\left(J_2\left(f\right), J_3\left(f\right)\right) = 0$ or in other words, the mapping $f$, as well as all mappings that are $\mathbb{P}\mathbf{SL}_2(\mathbb{C})$ - equivalent to $f$ are solutions of the third order differential equation

$$P_f\left(J_2, J_3\right) = 0.$$

**Theorem 2.** (1) *Two regular rational mappings $f$ and $g$ are $\mathbb{P}\mathbf{SL}_2(\mathbb{C})$ - equivalent if and only if their adjoint generating polynomial are proportional.*

(2) *Orbit of a regular rational mapping $f$ consists of the solution space of ordinary differential equation*

$$P_f\left(J_2, J_3\right) = 0.$$

(1) For linear fractional transformations $f\left(z\right) = \frac{az+b}{cz+d}$, $ad - bc = 1$ we have

$$J_2(f) = 4(c+d)^2$$

and therefore the corresponding differential equation has the following form:

$$u_1^{-3}\left(\left(z-u\right)u_2 + 2u_1\left(u_1+1\right)\right)^2 - 4(a+d)^2 = 0$$

with 2-dimensional space of solutions.

(2) For the Jukowski mapping $f\left(z\right) = \frac{z^2+1}{2z}$ we have

$$P_f\left(X, Y\right) = -4X + 3Y - 45$$

and therefore the corresponding differential equation has the following form

$$\left(z-u\right)^2\left(3u_1u_3 - 4u_2^2\right) + 2\left(z-u\right)u_1\left(u_1+1\right)u_2 +$$

$$+ u_1^2\left(2u_1-1\right)\left(u_1-2\right) = 0.$$

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# Automorphisms of the Kronrod-Reeb graphs of Morse functions on 2-sphere

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Let $M$ be a compact two-dimensional manifold and , $f \in C^\infty(M, \mathbb{R})$ is Morse function and $\mathit{\Gamma}_f$ its Kronrod-Reeb's graph. We denote the $O(f) = \{f \circ h \mid h \in D(M)\}$ orbit of $f$ with respect to the natural right action of the group of diffeomorphisms $D(M)$ on $C^\infty(M, \mathbb{R})$, and $S(f) = \{h \in D(M) \mid f \circ h = f\}$ is the stabilizer of this function. It is easy to show that each $h \in S(f)$ induces an automorphism of the graph $\mathit{\Gamma}_f$. Let also $S'(f) = S(f) \cap D_{\mathrm{id}}(M)$ be a subgroup of $D_{\mathrm{id}}(M)$, consisting of diffeomorphisms preserving $f$ and isotopic to identical mappings and $G_f$ be the group of automorphisms of the Kronrod-Reeb graph induced by diffeomorphisms belonging to $S'(f)$. This group is the key ingredient for calculating the homotopy type of the orbit $O(f)$. In the previous article, [4], the authors describe the structure of groups $G_f$ for Morse functions on all orientational surfaces, except for sphere and torus. In this paper we study the case $M = S^2$. In this situation $\mathit{\Gamma}_f$ is always a tree, and therefore all elements of the group $G_f$ have a common fixed $\mathrm{Fix(G_f)}$ subtree, which can be even from one point. The main result is to calculate the groups $G_f$ for all Morse functions $f : S^2 \to \mathbb{R}$ in which $\mathrm{Fix(G_f)}$ is not the point.

**Theorem 1.** *Let $f \in C^\infty(S^2, \mathbb{R})$ be Morse function on a sphere. Suppose that all elements of the group $G_f$ have a common fixed edge $E$. Let $x \in E$ be an arbitrary point and $A$ and $B$ is the closure of the connected components $S^2 \setminus p^{-1}(x)$. Then $A$ and $B$-double discs are invariant with respect to $S'(f)$, the restriction of $f|_A, f|_B$ are Morse functions and we have the following isomorphism:*

$$\phi : G_f \to G_{f_{|A}} \times G_{f_{|B}},$$

*is determined by the formula $\phi(\gamma) = (\gamma_{|\mathit{\Gamma}_A};\ \gamma_{|\mathit{\Gamma}_B})$.*

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# Algebra of block-symmetric analytic functions of bounded type

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In talk we will describe the spectrum of the algebra of block-symmetric analytic functions of bonded type on $\ell_1 \oplus \ell_\infty$. Let us denote by $\ell_1 \oplus \ell_\infty$ the space with elements $\begin{pmatrix} x \\ y \end{pmatrix} = \left( \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \ldots, \begin{pmatrix} x_m \\ y_m \end{pmatrix}, \ldots \right)$, where $(x_1, x_2, \ldots, x_n, \ldots) \in \ell_1$, $(y_1, y_2, \ldots, y_n, \ldots) \in \ell_\infty$. The space $\ell_1 \oplus \ell_\infty$ with norm

$$||(x, y)||_{\ell_1 \oplus \ell_\infty} = \sum_{i=1}^{\infty} |x_i| + \sup_{i \geq 1} |y_i|$$

is a Banach space.

Let us denote by $\mathcal{H}_{bvs}(\ell_1 \oplus \ell_\infty)$ the algebra of block-symmetric analytic functions of bounded type on $\ell_1 \oplus \ell_\infty$ and $\mathcal{M}_{bvs}(\ell_1 \oplus \ell_\infty)$ — the spectrum of this algebra.

In this talk we will describe the spectrum of the algebra of block-symmetric analytic functions of bounded type on $\ell_1 \oplus \ell_\infty$ and we will show that the spectrum of the algebra of block-symmetric analytic functions of bounded type on $\ell_1 \oplus \ell_\infty$ does not coincide of point evaluation functionals.

# Infinitesimal transformations of a symmetric Riemannian space of the first class

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P. A. Shirokov [1] found all irreducible symmetric Riemannian spaces $V_n$ of the first class. For $n = 4$, the metric tensor $g_{ij}(x)$ of such spaces in the Riemannian coordinate system with origin at the point $M_0$ $(x^h = 0)$ has the following form:

$$g_{ij}(x) = \underset{\circ}{g_{ij}} + \frac{1}{3} \left( b_{i\alpha} b_{j\beta} - b_{ij} b_{\alpha\beta} \right) x^\alpha x^\beta, \tag{1}$$

where

$$\left( \underset{\circ}{g_{ij}} \right) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad (b_{ij}) = \begin{pmatrix} \xi_1 & 0 & 1 & 0 \\ 0 & \xi_2 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{2}$$

$$(\xi_i = \pm 1, \ i = 1, 2)$$

In (2) $\underset{\circ}{g}_{ij}$ and $b_{ij}$, in the terminology of P. Shirokov, are values of the components of the 1st and 2nd fundamental tensors at the origin of the Riemannian coordinates.

For an arbitrary Riemannian space $V_n(x; g(x))$ S. M. Pokas' ([2]) introduced the concept of the second approximation space $\widetilde{V}_n^2(y; \widetilde{g}(y))$:

$$\widetilde{g}_{ij}(y) = \underset{\circ}{g}_{ij} + \frac{1}{3} \underset{\circ}{R}_{i\alpha\beta j}\, y^\alpha y^\beta \tag{3}$$

where $\underset{\circ}{g}_{ij} = g_{ij}(M_0)$, $\underset{\circ}{R}_{i\alpha\beta j} = R_{i\alpha\beta j}(M_0)$, and $M_0 \in V_n$ is an arbitrary point of source space.

A comparison of (3) and (1) leads to the conclusion that the symmetric Riemannian space of the first class $V_n$ is isometric to the space of the second approximation $\widetilde{V}_n^2$. Consequently, the Lie group of infinitesimal transformations $\widetilde{G}_r$ of the space $\widetilde{V}_n^2$ is isomorphic to the Lie group of infinitesimal transformations $G_r$ of a symmetric Riemannian space of the first class $V_n$.

Using the results of studies of infinitesimal transformations of the second approximation space $\widetilde{V}_n^2$, we obtain statements.

**Theorem 1.** *The infinitesimal conformal transformation of the second degree of the symmetric Riemannian space of the first class $V_n$ is necessarily homothetic.*

**Theorem 2.** *In the symmetric Riemannian space of the first class $V_n$, a Lie group of motions $G_8$ exists.*

The basis and structure of this group are found.

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# Unique solvability of the nonlocal problem with integral condition for nonhomogeneous differential equations of second order

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Let $H(([T_1, T_2] \cup [T_3, T_4]) \times \mathbb{R}_+)$ be a class of entirie function, $K_{L,M}$ be a class of quasi-polynomials of the form $f(t, x) = \sum_{i=1}^{n} \sum_{j=1}^{m} Q_{ij}(t, x) \exp[\alpha_i x + \beta_j t]$, where $Q_{ij}(t, x)$ are given polynomials, $L \subseteq \mathbb{C}$, $M \subseteq \mathbb{C}$ $\alpha_i \in L$, $\alpha_k \neq \alpha_l$, for $k \neq l$, $\beta_j \in M$, $\beta_k \neq \beta_l$, for $k \neq l$.

Each quasi-polynomial defines a differential operator $f\left(\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\nu}\right)$ of finite order on the class of certain function, in the form

$$\sum_{i=1}^{m}\sum_{j=1}^{m} Q_{ji}\left(\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\nu}\right)\exp\left[\alpha_i\frac{\partial}{\partial\lambda}+\beta_j\frac{\partial}{\partial\nu}\right]\Big|_{\lambda=0,\nu=0}.$$

In the strip $\Omega = \{(t,x)\in\mathbb{R}^2 : t\in(T_1,T_2)\cup(T_1,T_2), x\in\mathbb{R}_+\}$ we consider of the problem with integral conditios

$$\left[\frac{\partial^2}{\partial t^2}+a\left(\frac{\partial}{\partial x}\right)\frac{\partial}{\partial t}+b\left(\frac{\partial}{\partial x}\right)\right]U(t,x)=f(t,x), \tag{1}$$

satysfies nonlocal-integral conditions

$$\int_{T_1}^{T_2}U(t,x)dt+\int_{T_3}^{T_4}U(t,x)dt=0; \quad t\in[T_1,T_2]\cup[T_3,T_4]), \tag{2}$$

$$\int_{T_1}^{T_2}tU(t,x)dt+\int_{T_3}^{T_4}tU(t,x)dt=0; \tag{3}$$

where $a\left(\frac{\partial}{\partial x}\right)$, $b\left(\frac{\partial}{\partial x}\right)$ are differential expressions with entire functions $a(\lambda), b(\lambda)\neq$ *const*.

Solution of the problem (1), (2), (3) according to the differential-symbol method [1] is represented in the form

$$U(t,x)=f\left(\frac{\partial}{\partial\nu},\frac{\partial}{\partial\lambda}\right)\left\{G(t,\nu,\lambda)\exp[\lambda x]\right\}\Bigg|_{\lambda=\nu=0},$$

where $G(t,\nu,\lambda)$ is a solution of the problem:

$$\left[\frac{d^2}{dt^2}+a(\lambda)\frac{d}{dt}+b(\lambda)\right]G(t,\nu,\lambda)=\exp[\nu t],$$

$$\int_{T_1}^{T_2}G(t,\nu,\lambda)+\int_{T_3}^{T_4}G(t,\nu,\lambda)=0,$$

$$\int_{T_1}^{T_2}tG(t,\nu,\lambda)+\int_{T_3}^{T_4}tG(t,\nu,\lambda)=0.$$

This problem is a continuos works [2, 3].

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# Coexistence of Homoclinic Trajectories

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The nonperiodic trajectory of discrete dynamical systems is called $n$-homoclinic, if its $\alpha$- and $\omega$- limit sets coincide and are the same cycle of period $n$. For each point $x$ of this cycle we will call the subsequence of points of homoclinic trajectory which tends to it as $x$-subsequence.

We say that the homoclinic trajectory is one-sided, if for every point $x$ of this cycle every its $x$-subsequence tends to it from one side. If at least for one point $x$ of this cycle its $x$-subsequence tends to it from both sides, we will call the trajectory as two-sided homoclinic trajectory.

**Theorem 1.** *The ordering* $1 \triangleright 3 \triangleright 5 \triangleright 7 \triangleright \ldots \triangleright 2{\cdot}1 \triangleright 2{\cdot}3 \triangleright 2{\cdot}5 \triangleright \ldots \triangleright 2^2{\cdot}1 \triangleright 2^2{\cdot}3 \triangleright 2^2{\cdot}5 \triangleright \ldots$ *represents the coexistence of homoclinic trajectories of one-dimensional systems in the following sence.*

*If one-dimensional dynamical system has a one-sided $n$-homoclinic trajectory or two-sided $n/2$-homoclinic trajectory, then it also has a one-sided $m$-homoclinic trajectory or two-sided $m/2$-homoclinic trajectory for each $m \triangleleft n$.*

It has been proven the following theorem.

**Theorem 2.** *Every one-dimensional dynamical system that has a cycle of period $n \neq 2^i$ will also have a one-sided $n$-homoclinic trajectory.*

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# Properties of changing orientation homeomorphisms of the disk

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Denote by $D^2$ 2-dimensional disk. We will call a cell complex regularly embedded in $D^2$ if it is a subcomplex of a triangulation of $D^2$.

**Theorem 1.** *Let $K$ be a finite connected one-dimensional cell complex regularly embedded in the interior of $D^2$ and $K^0$ be the set of all vertices of $K$. Suppose there exists a homeomorphism $h\colon D^2 \to D^2$ reversing the orientation of $D^2$ such that $h(K) = K$ and $h(K^0) = K^0$. Then $h^2$ preserves every vertex of $K$ fixed and leaves every edge (open 1-cell) of $K$ invariant with preserved orientation.*

Denote by $\mathcal{D}(D^2)$ the group of $C^\infty$-diffeomorphisms of $D^2$. There is a natural right action of the group $\mathcal{D}(D^2)$ on the space of smooth functions $C^\infty(D^2, \mathbb{R})$ defined by the following rule: $(h, f) \longmapsto f \circ h$, where $h \in \mathcal{D}(D^2)$, $f \in C^\infty(D^2, \mathbb{R})$.

Thus, the *stabilizer* of $f$ with respect to the action

$$\mathcal{S}(f) = \{h \in \mathcal{D}(D^2) \mid f \circ h = f\}$$

consists of $f$-preserving diffeomorphisms of $D^2$.

Endow the space $\mathcal{D}(D^2)$ with Whitney $C^\infty$-topology and its subspace $\mathcal{S}(f)$ with the induced one. Denote by $\mathcal{S}_{\mathrm{id}}(f)$ the identity path component of $\mathcal{S}(f)$.

**Theorem 2.** *Let $f\colon D^2 \to \mathbb{R}$ be a Morse function. Suppose there exists $h \in \mathcal{S}(f)$ changing the orientation of $D^2$. Then exists diffeomorphism $g \in \mathcal{S}(f)$ such that $g = h$ on a neighborhood of $\partial D^2$ and $g^2 \in \mathcal{S}_{\mathrm{id}}(f)$.*

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# First Betti numbers of orbits of Morse functions on surfaces

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Let $\mathcal{G}$ be a minimal class of groups satisfying the following conditions:

(1) $1 \in \mathcal{G}$;
(2) if $A, B \in \mathcal{G}$, then $A \times B \in \mathcal{G}$;
(3) if $A \in \mathcal{G}$ and $n \geq 1$, then the wreath product $A \wr_n \mathbb{Z} \in \mathcal{G}$.

In other words a group $G$ belongs to the class $\mathcal{G}$ iff $G$ is obtained from trivial group by a finite number of operations $\times$, $\wr_n\mathbb{Z}$. It is easy to see that every group $G \in \mathcal{G}$ can be written as a word in the alphabet

$$\mathcal{A} = \{1, \mathbb{Z}, (,), \times, \wr_2, \wr_3, \wr_4, \dots\}.$$

We will call such word a *presentation* of the group $G$ in the alphabet $\mathcal{A}$. Evidently, the presentation of a group is not uniquely determined.

Denote by $Z(G)$ and $[G,G]$ the center and the commutator subgroup of $G$ respectively.

**Theorem 1.** *Let $G \in \mathcal{G}$, $\omega$ be an arbitrary presentation of $G$ in the alphabet $\mathcal{A}$, and $\beta_1(\omega)$ be the number of symbols $\mathbb{Z}$ in the presentation $\omega$. Then there are the following isomorphisms:*

$$Z(G) \cong G/[G,G] \cong \mathbb{Z}^{\beta_1(\omega)}.$$

*In particular, the number $\beta_1(\omega)$ depends only on the group $G$.*

The groups from the class $\mathcal{G}$ appear as fundamental groups of orbits of Morse functions on surfaces. Let $M$ be a compact surface and $\mathcal{D}$ be the group of $C^\infty$-diffeomorphisms of $M$. There is a natural right action of the group $\mathcal{D}$ on the space of smooth functions $C^\infty(M,\mathbb{R})$ defined by the rule: $(f,h) \longmapsto f \circ h$, where $h \in \mathcal{D}$, $f \in C^\infty(M,\mathbb{R})$.

Let $\mathcal{O}(f) = \{f \circ h \mid h \in \mathcal{D}\}$ be the *orbit* of $f$ under the above action. Endow the spaces $\mathcal{D}$, $C^\infty(M,\mathbb{R})$ with Whitney $C^\infty$-topologies. Let $\mathcal{O}_f(f)$ denote the path component of $f$ in $\mathcal{O}(f)$.

A map $f \in C^\infty(M,\mathbb{R})$ will be called *Morse* if all its critical points are non-degenerate. Homotopy types of stabilizers and orbits of Morse functions were calculated in a series of papers by Sergiy Maksymenko [3], [2], Bohdan Feshchenko [4], and Elena Kudryavtseva [1]. As a consequence of Theorem 1 we get the following.

**Corollary 2.** *Let $M$ be a connected compact oriented surface distinct from $S^2$ and $T^2$, $f$ be a Morse function on $M$, $G = \pi_1\mathcal{O}_f(f) \in \mathcal{G}$, $\omega$ be an arbitrary presentation of $G$ in the alphabet $\mathcal{A}$, and $\beta_1(\omega)$ be the number of symbols $\mathbb{Z}$ in the presentation $\omega$. Then the first integral homology group $H_1(\mathcal{O}(f),\mathbb{Z})$ of $\mathcal{O}(f)$ is a free abelian group of rank $\beta_1(\omega)$:*

$$H_1(\mathcal{O}(f),\mathbb{Z}) \simeq \mathbb{Z}^{\beta_1(\omega)}.$$

*In particular, $\beta_1(\omega)$ is the first Betti number of $\mathcal{O}(f)$.*

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# Optimal Morse flows on 2-manifolds with the boundary


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We consider optimal Morse flows on 2-manifolds, i.e. Morse-Smale flows without closed orbit and minimal numbers of fixed points and separatrixes. We also suppose that all fixed points belong to the boundary. It is proved that the flow be optimal if it has a single sink and a single source. We describe all possible topological structures of such flows on a 2-disk, a Mobius strip, a torus and a Klein bottle. On these surfaces, there are one, one, two and four structures, respectively.

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# On orbit braids

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Let $M$ be a connected topological manifold of dimension at least 2 with an effective action of a finite group $G$. Associating with the orbit configuration space $F_G(M,n), n \geq 2$ of the $G$-manifold $M$, we try to upbuild the theoretical framework of orbit braids in $M \times I$ where the action of $G$ on $I$ is trivial, which contains the following contents.

We introduce the orbit braid group $\mathcal{B}_n^{orb}(M,G)$, and show that it is isomorphic to a group with an additional endowed operation (called the extended fundamental group), formed by the homotopy classes of some paths (not necessarily closed paths) in $F_G(M,n)$, which is an essential extension for fundamental groups. The orbit braid group $\mathcal{B}_n^{orb}(M,G)$ is large enough to contain the fundamental group of $F_G(M,n)$ and other various braid groups as its subgroups. Around the central position of $\mathcal{B}_n^{orb}(M,G)$, we obtain five short exact sequences weaved in a commutative diagram. We also analyze the essential relations among various braid groups associated to those configuration spaces $F_G(M,n), F(M,n)$, and $F(M/G,n)$. We finally consider how to give the presentations of orbit braid groups in terms of orbit braids as generators. We carry out our work by choosing $M = \mathbb{C} \approx \mathbb{R}^2$ with typical actions of $\mathbb{Z}_p$ and $(\mathbb{Z}_2)^2$. We obtain the presentations of the corresponding orbit braid groups, from which we see that the generalized braid group $Br(B_n)$ (introduced by Brieskorn) actually agrees with the orbit braid group $\mathcal{B}_n^{orb}(\mathbb{C}\setminus\{0\},\mathbb{Z}_2)$ and $Br(D_n)$ is a subgroup of the orbit braid group $\mathcal{B}_n^{orb}(\mathbb{C},\mathbb{Z}_2)$. This is a joint work with Hao Li and Fengling Li

## Quotient spaces and their authomorphism spaces

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Let $Y$ be a topological space. Say that two points $y, z \in Y$ are $T_2$*-disjoint* in $Y$ if they have disjoint neighborhoods. Denote by $\mathrm{hcl}(y)$ the set of all $z \in Y$ that are *not* $T_2$*-disjoint from* $y$. Then $z \in \mathrm{hcl}(y)$ if and only if each neighborhood of $z$ intersects each neighborhood of $y$. We will call $\mathrm{hcl}(y)$ the *Hausdorff closure* of $y$.

Thus the relation $y \in \mathrm{hcl}(z)$ is reflexive and symmetric, however in general it is not transitive.

Say that a point $y \in Y$ is *special* whenever $\mathrm{hcl}(y) \setminus y \neq \emptyset$, so there are points that are not $T_2$-disjoint from $y$. We will denote by $\mathrm{Spec}(Y)$ the set of all special points of $Y$.

Let $X$ be a topological space, $\Delta = \{\omega_y \mid y \in Y\}$ be a partition of $X$, and $p : X \to Y$ be the natural quotient map such that $p(x) = y \in Y$ iff $x \in \omega_y$. Endow $Y$ with the corresponding quotient topology with respect to $p$,

Let $\mathcal{E}(Y) = C(Y, Y)$ be the monoid of all continuous maps $Y \to Y$ with respect to the natural composition of maps, and $\mathcal{E}(X, \Delta)$ be the monoid of all continuous maps $h : X \to X$ preserving $\Delta$ in the sense that $h(\omega)$ is contained in some leaf of $\Delta$ for each $\omega \in \Delta$. Denote by $\mathcal{H}(Y) \subset \mathcal{E}(Y)$ and $\mathcal{H}(X, \Delta) \subset \mathcal{E}(X, \Delta)$ the subgroups consisting of homeomorphisms.

It follows that each $h \in \mathcal{E}(X, \Delta)$ induces a map $\psi(h) : Y \to Y$ making commutative the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{h} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{\psi(h)} & Y \end{array}$$

Since $Y$ is endowed with quotient topology with respect to $p$, it follows that

$$\psi : \mathcal{E}(X, \Delta) \to \mathcal{E}(Y) \tag{1}$$

is a *homomorphism of monoids*, that is $\psi(h_1 \circ h_2) = \psi(h_1) \circ \psi(h_2)$ for all $h_1, h_2 \in \mathcal{E}(X, \Delta)$ and $\psi(\mathrm{id}_X) = \mathrm{id}_Y$. This implies that $\psi$ induces the homomorphism $\psi : \mathcal{H}(X, \Delta) \to \mathcal{H}(Y)$ between the corresponding homeomorphism groups.

**Theorem 1.** *Suppose that*

i) *$X$ is a locally compact Hausdorff topological space,*
ii) *$Y$ is a $T_1$-space, i.e., each element of $\Delta$ is closed;*
iii) *the projection $p : X \to Y$ is an open map;*
iv) *the set $\mathrm{Spec}(Y)$ of special points of $Y$ is locally finite.*

*Then the homomorphism (1) $\psi : \mathcal{E}(X, \Delta) \to \mathcal{E}(Y)$ is continuous with respect to the corresponding compact open topologies.*

*In particular so is the induced homomorphism $\psi : \mathcal{H}(X, \Delta) \to \mathcal{H}(Y)$.*

**Corollary 2.** *Suppose the conditions of previous theorem are fullfilled and $X$ is connected. Then we have a well-defined homomorphism $\psi_0 : \pi_0\mathcal{H}(X, \Delta) \to \pi_0\mathcal{H}(Y)$ of the corresponding mapping class groups.*

## Realization problems for Reeb graphs and epimorphisms onto free groups

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The Reeb graph $\mathcal{R}(f)$ of a Morse function $f$ on a manifold is obtained by contracting connected components of its level sets. Sharko and Masumoto–Saeki showed that each graph with the so-called good orientation is the Reeb graph of a function on a closed surface. In this talk we focus on the problem of realization of a graph as the Reeb graph of a function on a given manifold $M$. In particular, it turns out that the maximal number of cycles among all Reeb graphs of functions on $M$ is equal to the corank of $\pi_1(M)$, i.e. the maximal rank $r$ for which there is an epimorphism $\pi_1(M) \to F_r$ onto the free group of rank $r$. It leads to the natural problem of representing any such epimorphism by the homomorphism induced on fundamental groups by the quotient map $M \to \mathcal{R}(f)$ for a Morse function $f$. We describe connections between Reeb graphs, epimorphisms onto free groups and systems of nonseparating 2-sided submanifolds. This allows us to study algebraic properties of epimorphisms onto free groups, such as their equivalence classes or ranks of maximal epimorphisms. For example, using theorems on topological conjugation of simple Morse functions on orientable (Kulinich, Sharko) or non-orientable surfaces (Lychak–Prishlyak), we may repeat some results on equivalence classes of epimorphisms onto free groups for surface groups provided by Grigorchuk–Kurchanov–Zieschang.

The results are from joint work with Wacław Marzantowicz.

## The use of K-theory in high energy physics

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Modern high-energy physics associated with energies up to 14 teV is the theory of D-brane and superstrings, [1]. The mathematical apparatus of such a theory is algebraic geometry and topological algebra, which uses the theory of derived categories.

We will consider the case of supersymmetric string theory extended to N = (2,2) superconformal field theory with target manifold X - Calabi–Yau threefold. With this superconformal string field theory is associated central charge. On the end of the string for D-brane system lives a Ramon-Ramon (RR) charge which takes values in a two-dimensional quantum Hilbert space.

The information of RR charge can be decoded in the category of D-branes, which is considered as derived category of coherent sheaves, D(X) over X - a topological space. An open string from the D-brane associated to the locally-free sheaf E to another D-brane given by the locally-free sheaf F is given by an element of the group $Ext^q(E,F)$ which is Hilbert space. D-brane/anti-D-brane annihilation which is built in the derived category map the derived category to K-theory language for D-branes [2]. In the case of the twisted bundle D-brane charge takes values in a certain twisted version of K-theory with special type of sections as a Hilbert space [3]. With different types of the structure group of the twisted bundles are connected different K-theory groups.

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# A note on cohomology of locally trivial Lie groupoids on triangulated spaces


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Mishchenko and Oliveira proved that, for each transitive Lie algebroid defined on a compact triangulated manifold, its Lie algebroid cohomology and piecewise smooth cohomology are isomorphic. Based in that isomorphism, it is proved that the Rham cohomology of a locally trivial Lie groupoid $G$ on a smooth manifold $M$ is isomorphic to the piecewise Rham cohomology of $G$, in which $G$ and $M$ are manifolds without boundary and $M$ is smoothly triangulated by a finite simplicial complex $K$ such that, for each simplex $\Delta$ of $K$, the inverse images of $\Delta$ by the source and target mappings of $G$ are transverses submanifolds in the ambient space $G$.

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# Symplectic Floer theory


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I plan to explain symplectic Floer theory for non-specialists.

After giving basic ideas in the theory, I speak on examples and applications to symplectic geometry.

# $Z$-knotted triangulations of surfaces


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*Joint work with Adam Tyc (IM PAN, Warsaw)*

Petrie polygons are well-known objects described by Coxeter. These are skew polygons in regular polyhedra such that any two consecutive edges, but not three, are on the same face.

Analogs of Petrie polygons for graphs embedded in surfaces are called *zigzags*. They have many applications, for example, they are successfully exploited to enumerate all combinatorial possibilities for fullerenes. The case when an embedded graph has a single zigzag is closely connected to Gauss code problem. An embedded graph with a unique zigzags is said to be *z-knotted*.

We investigate zigzags in triangulations of closed (not necessarily orientable) surfaces and show that every such triangulation admits a $z$-knotted shredding.

Our main tool is the concept of $z$-monodromy.

We describe all possibilities for $z$-monodromies of faces in triangulations: there are precisely 7 types of $z$-monodromies and 4 types corresponding to the $z$-knotted case.

# Right-angled polytopes, hyperbolic manifolds and torus actions

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A combinatorial 3-dimensional polytope $P$ can be realised in Lobachevsky 3-space with right dihedral angles if and only if it is simple, flag and does not have 4-belts of facets. This criterion was proved in the works of Pogorelov and Andreev of the 1960s. We refer to combinatorial 3-polytopes admitting a right-angled realisation in Lobachevsky 3-space as Pogorelov polytopes. The Pogorelov class contains all fullerenes, i.e. simple 3-polytopes with only 5-gonal and 6-gonal facets.

There are two families of smooth manifolds associated with Pogorelov polytopes. The first family consists of 3-dimensional small covers of Pogorelov polytopes $P$, also known as hyperbolic 3-manifolds of Loebell type. These are aspherical 3-manifolds whose fundamental groups are certain finite abelian extensions of hyperbolic right-angled reflection groups in the facets of $P$. The second family consists of 6-dimensional quasitoric manifolds over Pogorelov polytopes. These are simply connected 6-manifolds with a 3-dimensional torus action and orbit space $P$. Our main result is that both families are cohomologically rigid, i. e. two manifolds $M$ and $M'$ from either family are diffeomorphic if and only if their cohomology rings are isomorphic. We also prove that a cohomology ring isomorphism implies an equivalence of characteristic pairs; in particular, the corresponding polytopes $P$ and $P'$ are combinatorially equivalent. This leads to a positive solution of a problem of Vesnin (1991) on hyperbolic Loebell manifolds, and implies their full classification.

Our results are intertwined with classical subjects of geometry and topology such as combinatorics of 3-polytopes, the Four Colour Theorem, aspherical manifolds, a diffeomorphism classification of 6-manifolds and invariance of Pontryagin classes. The proofs use techniques of toric topology.

This is a joint work with V. Buchstaber, N. Erokhovets, M. Masuda and S. Park.

# On configuration spaces of $k$ thick particles in a rectangle

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Let $Q$ be a rectangle $[0,a] \times [0,b] \subset \mathbb{R}^2$ with $0 < a \leq b$. and let $F_k(Q,\varepsilon)$ denote *the configuration space of $k$ squares in $Q$ with all sides of the same length $2\varepsilon$, which do not overlap each other.* We consider $2\varepsilon$-squares in the rectangle $Q$ as the $\varepsilon$-discs in the *max* norm metric $d(x,y)$. It is allowable that different squares can have common boundary points and the boundaries of squares can intersect the boundary $\partial Q$ along some intervals. Therefore $F_k(Q,\varepsilon)$ consists of ordered $k$-tuples $(u_1,\dots,u_k)\in Q^k$ such that for any $i,j,i\neq j$, $d(u_i,u_j)\geq 2\varepsilon$ and for each $u_i$ we have $d(u_i,\partial Q)\geq\varepsilon$.

In this work we study critical values of the parameter $\varepsilon$ with respect to change of topology of $F_k(Q,\varepsilon)$. In particular, we show that for values of the parameter $\varepsilon$ enough small $F_k(Q,\varepsilon)$ is connected. We also discuss the problem for which parameters $\varepsilon$ the space $F_k(Q,\varepsilon)$ is aspherical.

## Morse functions and Morse flows on low-dimensional manifolds with the boundary

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We consider functions and flows on 2 and 3-dimensional manifolds with the boundary, all critical (fixed) points of which belong to the boundary of the manifold. In this case there is the analogue of Morse functions. They are functions which have only non-degenerated critical points and their restrictions to the boundary have the same critical points that are also non-degenerated. There is the neighborhood of each of these points in such a way that the function $f$ takes one of the following forms, [1]:

$$f(x,y)=-x_1^2-...-x_i^2+x_{i+1}^2+...+x_{n-1}^2\pm x_n, x_n\geq 0.$$

Besides in the case of isolated singular points on 2-manifold, the function can be represented in the form $f(x,y)=Re(x+iy), y\geq 0$ for some appropriate local coordinates $(x,y)$.

Gradient-like flows of Morse functions in general position are Morse flows (Morse-Smale flows without orbits). On manifolds $M$ with boundary $\partial M$ it is a flow $X$ which satisfies the following conditions:

1) the set of nonwandering points $\Omega(X)$ has finite number of orbits and all of them are hyperbolic,

2) if $u,v\in\Omega(X)\cap\mathrm{Int}M$ then unstable manifold $W^u(u)$ is transversal to stable manifold $W^s(v)$,

3) for $u,v\in\Omega(X)$, if $x\in M$ is a point of nontransversal intersection of $W^u(u)$ with $W^s(v)$ then $x\in\partial M$ and either $u$ or $v$ is a singularity of $X$ [2].

Morse flows on the surface with boundary can have four types of fixed points on the boundary: 1) a source, 2) a sink, 3) a-saddle and 4) b-saddle. The topological structure of such flows is determined by the separatrixes.

There is a flow with one singular point for any connected surface with a connected boundary. Separatrix breaks neighborhood of this point into the corners that can have four types: 1) hyperbolic, 2) eliptic 3) sources and 4) sink. Location separatrix and specifying types of angles determines the structure of such flows.

In dimension 3 generalized Heegaard diagrams [3] can be used to determine the structure of Morse flows.

Let M be a smooth compact 3-manifold with boundary.

We construct a diagram of Morse flow, which has the form of a surface with a boundary and two sets of arcs and circles embedded in it. The surface $F$ is the boundary of the regular neighborhood of the union of the following integral manifolds:

1) sources and stable manifolds of singular points of index 1 in the interior of the manifold;

2) sources on the boundary, which are sources on doubling;

3) stable manifolds of saddle points and boundary sources, which are points of index 1 on doubling.

On the surface F, select the following sets of nested arcs and circles:

1) circles u, which are intersections of unstable manifolds of interior singular points of index 1 with surface F;

2) the arcs U are intersections of F with unstable manifolds of saddle singular points of the edge, which are points of index 1 on doubling;

3) circles v, which are intersections of stable varieties of interior singular points of index 2 with surface F;

4) arc V - intersections of F with stable manifolds of saddle singular points of the edge, which are points of index 2 at doubling.

If we do a surgery of F along u and U, we obtain the 2-sphere with holes. Its boundary component correspond to the boundary singular points - one source and several saddles. We do first marking of arc of F that form boundary source component by 0 and others by 1. Surgery of F along v and V gives another marking of the arcs of the boundary by 0 and 2. Denote by w the framing that corresponds the sum of first and second marking to each arc of the boundary.

By a *Morse flow diagram* on a three-dimensional manifold with a boundary we will call the set

$$(F, u, U, v, V, w)$$

consisting of a surface with the boundary, a set of circles and arcs embedded in it as above and a framing.

Two Morse flow diagrams are said to be isomorphic if there is a surface homeomorphism that maps the sets of arcs and circles into sets of arcs and circles of the same type and preserve framing.

**Theorem 1.** *Two Morse flows on 3-manifold with a boundary are topologically trajectory equivalent if and only if their diagrams are isomorphic.*

A Morse flow diagrams can be used to classify Morse functions on 3-manifolds.

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# Topological properties of Morse-Smale flows on a compact surface with boundary

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In this paper we consider the Morse flows [1] (Morse-Smale flows without closed orbits) on the compact surfaces with boundary. There was constructed a complete topological invariant of these flows – an equipped three-colored graph.

The graph $T$ will be called *three-color graph*, if all its vertices have a degree not greater than 3, and edges are painted in three colors, so that edges of different colors converge at each vertex. Colors are denoted by the letters $s, t, u$, [2, 3]. The vertices of three-colored graph correspond to the standard areas on the surface, that look like a curvilinear triangle or quadrilateral.

There were found conditions in which a three-colored graph generates a flow.

**Theorem 1.** *For a connected tricolor graph having the following properties:*

1) *each edge of the graph is marked with one of the three letters: $s, t, u$, and each vertex is white or black;*
2) *two edges of the same type can not come out from each vertex;*
3) *for each black inner vertex there is a su-cycle of length 4 that contains it;*
4) *if two black vertices are connected by a u-or s-edge and one of them is bounded, then the other will be bound;*
5) *each white vertex is internal. And if it is connected to the black vertex u-edge (s-edge), then this black vertex will be the limit.*

*there exists a Morse flow on a connected surface with a boundary, the three-color graph of which is a given graph.*[1]

The number of topologically non-equivalent flows with 2, 3, 4, and 5 standard areas was calculated. For each of them, the surface on which this flow is set is determined.

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# $S^1$-Bott functions on manifolds

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Let $M^n$ be a compact closed manifold of dimension at least 3. With V. Sharko we studied the $S^1$-Bott functions on $M^n$. Separately, we investigated $S^1$-invariant Bott functions on $M^{2n}$ with semifree circle action that have finitely many fixed points. Our aim was to find the exact values of the minimum numbers of singular circles of some indices of $S^1$-invariant Bott functions on $M^{2n}$. I shall describe our results and also state some open problems and conjectures.

# On topology of spaces of persistence diagrams

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The Topological Data Analysis (TDA) provides metric and topological tools for investigation of large data arrays. Usually, the sets of data possess a structure of filtered topological space and can be investigated be means of persistent homology.

In turn, the persistence homology classes can be represented by the persistence diagrams indication the "birth" and "death" of every such class (see, e.g., [1]–[3]).

The persistence diagrams are therefore important objects in the TDA. The set of persistence diagrams can be endowed with various metrics, in particular, the bottleneck metric. We also endow the set of persistence diagrams with a nonmetrizable topology such that the obtained topological space is homeomorphic to the countable direct limit of infinite tower of Euclidean spaces $\mathbb{R}^\infty$. Also, completions of the metric spaces of persistence diagrams are considered.

The (completed) spaces of persistence diagrams are infinite-dimensional. The aim of the talk is to apply methods of infinite-dimensional topology (in particular, characterization theorems for infinite-dimensional manifolds) to description of topology of these spaces. In particular, we prove that some of these spaces are homeomorphic to the pre-Hilbert spaces of finite sequences.

Also, we show that the complement of the space of persistence diagrams is locally homotopy negligible in its completion. This naturally leads to the problem whether the completed space of persistence diagrams is an absolute retract.

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# Analogue of Whitney trick for eliminating multiple intersections

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The Whitney trick for cancelling *double* intersections is one of the main tools in the topology of manifolds. Generalization of the Whitney trick to *multiple* intersections was 'in the air' since 1960s.

However, only in this century they were stated, proved and applied to obtain interesting results.

I shall describe the ground-breaking work [MW15] (see also the survey [Sk16]) and its generalizations to *codimension 2* [AMS+] and to the case when *general position multiple intersections have positive dimension* [MW16, Sk17].

These were most difficult steps in recent counterexamples to the topological Tverberg conjecture (for which papers by M. Özaydin, M. Gromov, P. Blagojević, F. Frick, G. Ziegler, I. Mabillard and U. Wagner are important, see the survey [Sk16]) and in stronger counterexamples [AMS+, AKS].

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# The commutator of Sylow subgroups of alternating and symmetric groups, these minimal generating sets

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We consider the commutator of Sylow subgroups of an alternating group and research its minimal generating sets. The commutator width of a group $G$, denoted by $cw(G)$ [1], is the maximum of commutator lengths of elements of its derived subgroup $[G, G]$. The size of minimal generating sets of the Sylow 2-subgroup $Syl_2 A_{2^k}$ of $A_{2^k}$ is found. The commutator width of Sylow 2-subgroups of the alternating group $A_{2^k}$, symmetric group $S_{2^k}$ and $C_p \wr B$ are equal to 1. We prove that the commutator width [1] of an arbitrary element of a permutational wreath product of cyclic groups $C_{p_i}$, $p_i \in \mathbb{N}$, is 1. As it has been proven in [2] there are subgroups $G_k$ and $B_k$ in the automorphisms group $AutX^{[k]}$ of the restricted binary rooted tree such that $G_k \simeq Syl_2 A_{2^k}$ and $B_k \simeq Syl_2 S_{2^k}$, respectively.

**Theorem 1.** *An element $(g_1, g_2)\sigma \in G'_k$, where $\sigma \in S_2$ if and only if $g_1, g_2 \in G_{k-1}$ and $g_1 g_2 \in B'_{k-1}$.*

**Lemma 2.** *For any group $B$ and integer $p \geq 2$ the following inequality is true:*

$$cw(B \wr C_p) \leq \max(1, cw(B)).$$

**Corollary 3.** *For a prime $p > 2$ and $k > 1$ the commutator widths of $Syl_p(A_{p^k})$ and of $Syl_p(S_{p^k})$ are equal to 1.*

**Theorem 4.** *Elements of $Syl_2 S'_{2^k}$ have the following form:*

$$\{[f, l] \mid f \in B_k, l \in G_k\} = \{[l, f] \mid f \in B_k, l \in G_k\}.$$

Moreover, we get a more general result about the commutator width for a finite wreath product of finite cyclic groups.

**Corollary 5.** *If $W = C_{p_k} \wr \ldots \wr C_{p_1}$ then for $k \geq 2$ we have $cw(W) = 1$.*

**Theorem 6.** *The commutator width of the group $Syl_2 A_{2^k}$ is equal to 1 for $k \geq 2$.*

**Theorem 7.** *A commutator of $G_k$ has the form ${G'}_k \simeq G_{k-1} \odot G_{k-1}$, where $\odot$ is the subdirect product. The order of ${G'}_k$ is equal to $2^{2^k-k-2}$. A second commutator of $G_k$ has a structure ${G''}_k \simeq {G'}_{k-1} \odot {G'}_{k-1}$ , ${G''}_k \simeq (G_{k-2} \odot G_{k-2}) \odot (G_{k-2} \odot G_{k-2})$,*

**Proposition 8.** *The subgroup $(Syl_2 A_{2^k})'$ has a minimal generating set of $2k - 3$ generators.*

A minimal generating set of $Syl'_2(A_8)$ consists of 3 generators: $(1,3)(2,4)(5,7)(6,8)$, $(1,2)(3,4)$, $(1,3)(2,4)(5,8)(6,7)$. In addition, $Syl'_2(A_8) \simeq C_2^3$ and it is an elementary abelian 2-group of order 8.

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# Singularities of curves with two-parameter families of ideals


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We consider the study of ideals of commutative rings, in particular, the question about parametrization of classes of ideals is of current importance for the modern algebra.

Recall that a plane curve singularity over a field $k$ is a $k$-algebra of the form

$$R = k\,[[x, y]]/(f).$$

It is called one branch if $R$ has no zero divisors.

Till now almost nothing was known about the curve singularities with at most $m$-parametric families of ideals if $m > 1$, in particular, how it relates with the Arnold's classification of singularities.

Sufficient and necessary conditions of possessing of one branch curve singularities at most 2-parameter families of ideals were researched.

**Theorem 1.** *If $R$ is one branch singularity. Then the following conditions are equivalent:*

1) *$R$ has as maximum two parametric family of ideals.*
2) *If char $k \neq 2$, then $R$ dominates one of the following singularities:*

$$E_{30},\ E_{32}, W_{24}, W^{\#}_{2\,,\,*}, N_{30}, N_{20},\ N_{24},\ N_{28};$$

2a) *If char $k = 2$, then $R$ dominates one of the following singularities:*

$$E_{30},\ E_{32}, W_{18}, W^{\#}_{1\,,\,*}, N_{20}, N_{24}.$$

Thus, it is proved sufficient and necessary conditions for a one-branch curve singularity S has at most two-parameter families of ideals.

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# About multiary webs

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In [1] binary $k$-webs are investigated. In [2] algebraic aspects of web geometry, namely its connections with the quasigroup and loop theory, the theory of local differential quasigroups and loops, and the theory of local algebras are discussed. Here, we give a definition of $n$-ary $k$-web and prove some properties.

Let $Q$ be an arbitrary finite or infinite set. A mapping $f : Q^n \to Q$ is called an *$n$-ary operation* defined on $Q$.

A set $\{f_1, \ldots, f_n\}$ of $n$-ary operations defined on $Q$ is called orthogonal if for arbitrary elements $a_1$, ..., $a_n \in Q$ the system

$$\begin{cases} f_1(x_1, \ldots, x_n) = a_1, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ f_n(x_1, \ldots, x_n) = a_n \end{cases}$$

has a unique solution.

**Definition 1.** A $k$-web $W$ of the arity $n$ consists of a set $\mathcal{P}$ of points and a subset $\mathcal{L}$ of the power set of $\mathcal{P}$ whose elements are called lines. In $\mathcal{L}$, there are $k$ subsets called pencils ($k > n$) such that the following axioms hold:

(W1) Each line belongs to just one of the pencils of $W$.
(W2) Each point belongs to just one line from each pencil.
(W3) Any $n$ lines from distinct pencils have exactly one point in common, whereas lines from the same pencil are disjoint.

**Lemma 2.** *There is a one-to-one correspondence between: 1) the points of a line and the lines of a pencil; 2) the lines of two arbitrary pencils; 3) the points of two arbitrary lines.*

Thus, the point set of a line and the line set of a pencil have the same cardinality $m$ called *order* of the web. Since the lines of a pencil are disjoint, then the cardinal number of $\mathcal{P}$ is $m^n$.

Let $Q$ be a set of the order $m$, i.e., $|Q| = m$. Let us bijectively label all lines of each pencil by elements of the set $Q$ and all points of the web by elements of the set

$Q^n$. An expression $P(a_1, \ldots, a_n)$ means that the point $P$ is labeled by $(a_1, \ldots, a_n)$ which is also called *coordinates* of $P$.

For each pencil $\mathcal{L}_i$, we define an $n$-ary operation $f_i$ on the set $Q$: $f_i(a_1, \ldots, a_n) = a$ if the point with the coordinates $(a_1, \ldots, a_n)$ belongs to the line from $\mathcal{L}_i$ which is labeled by the element $a \in Q$.

The obtained system $\Sigma := \{f_1, \ldots, f_k\}$ of operations are called *coordinate system of operations* or *coordinate operation system* (COS) of the web.

**Theorem 3.** *A set of $k$ $n$-ary operations ($k > n$) is a coordinate operation system of an $n$-ary $k$-web if and only if it is orthogonal.*

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# Topology of the set of factorable almost periodic matrix functions

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A classical Wiener-Hopf factorization of (matrix) functions defined on the unit circle $\mathbb{T}$ or the real line $\mathbb{R}$ is their representation as a product of three factors, the left/right analytic and invertible inside/outside $\mathbb{T}$ (respectively, in the upper/lower half-pane), and the diagonal middle factor with the diagonal entries of some special form.

For several classes of functions, scalar or matrix valued, the factorablity is equivalent to invertibility. These classes include the Wiener algebra $W$ of functions with absolutely convergent Fourier series, continuous (matrix) functions, etc.

The same is true for the (scalar) almost periodic functions with absolutely convergent Bohr-Fourier series (the algebra $APW$), or the algebra $AP$ of all Bohr almost periodic functions.

This property is lost, however, in transition to **matrix**-valued $AP$ or $APW$ functions. Moreover, the respective factorability criteria are presently not know even in the case of 2-by-2 triangular matrix-functions in these classes.

The configuration of the set $GLF$ of all factorable $AP$ or $APW$ matrix functions within the respective group $GL$ of invertible matrix-functions is therefore a non-trivial issue.

As was established jointly with A. Brudnyi and L. Rodman, there are infinitely many pathwise connected components of $GL$ not intersecting with $GLF$, and even with the closed subgroup of $GL$ generated by it.

# Extension of continuous operators on $C_b(X, E)$ with the strict topology

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In the paper [2] Nowak has developed the theory of continuous linear operators on the space $C_b(X, E)$ of bounded continuous functions $f : X \to E$, where $X$ is a completely regular Hausdorff space and $E$ is a Banach space. Then the space $C_b(X, E)$ is equipped with the strict topology $\beta$. Recall that $\beta$ is generated by the family of the seminorms:

$$p_v(f) := \sup_{t \in X} |v(t)| \, \|f(t)\|_E \quad \text{for} \quad f \in C_b(X, E),$$

where $v : X \to \mathbb{R}$ is a bounded function such that for every $\varepsilon > 0$, $\{t \in X : |v(t)| \geq \varepsilon\}$ is a compact subset of $X$. For $X$ being a locally compact space $\beta$ coincides with the original strict topology that was introduced in 1958 by Buck [1]. The Riesz Representation Theorem for continuous linear operators $T : C_b(X, E) \to F$ was obtained, where $F$ is a Banach space (see [2, Theorem 3.1]).

Let $\mathcal{L}^\infty(\mathcal{B}o, E)$ stand for the set of all bounded strongly $\mathcal{B}o$-measurable functions $g : X \to E$. Then $\mathcal{L}^\infty(\mathcal{B}o, E)$ can be equipped with the natural mixed topology $\gamma_{\mathcal{L}^\infty(\mathcal{B}o,E)}$. Note that if $X$ is separable (resp. $E$ is separable), then

$$C_b(X, E) \subset \mathcal{L}^\infty(\mathcal{B}o, E).$$

The aim of my talk is to present some results concerning the problem of extension of different classes of $(\beta, \|\cdot\|_F)$-continuous linear operators $T : C_b(X, E) \to F$ to the corresponding classes of $(\gamma_{\mathcal{L}^\infty(\mathcal{B}o,E)}, \|\cdot\|_F)$-continuous linear operators $\overline{T} : \mathcal{L}^\infty(\mathcal{B}o, E) \to F$.

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# Chebotarev link is stably generic

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The analogy between knots and prime numbers, or 3-manifolds and the ring of integers of number fields, was initially pointed out by B. Mazur in [3], and developed by Kapranov, Reznikov, and Morishita in a systematic way (cf.[7]). In their study called *Arithmetic Topology*, an important problem is *to find a nice analogue of the set of all prime numbers.*

McMullen [5] established a version of the Chebotarev density theorem in which number fields are replaced by 3-manifolds, answering to Mazur's question proposed in [4]:

**Definition 1** (Chebotarev law)**.** Let $(K_i) = (K_i)_{i\in\mathbb{N}_{>0}}$ be a sequence of disjoint knots in a 3-manifold $M$. For each $n \in \mathbb{N}_{>0}$ and $j > n$, we put $L_n = \cup_{i\leq n} K_n$ and denote the conjugacy class of $K_j$ in $\pi_1(M - L_n)$ by $[K_j]$. We say that $(K_i)$ *obeys the Chebotarev law* if

$$\lim_{\nu\to\infty} \frac{\#\{n < j \leq \nu \mid \rho([K_j]) = C\}}{\nu} = \frac{\#C}{\#G}$$

holds for any $n \in \mathbb{N}_{>0}$, any surjective homomorphism

$$\rho : \pi_1(M - L_n) \to G$$

to any finite group, and any conjugacy class $C \subset G$. (The left hand side is *the natural density* of $K_i$'s with $\rho([K_j]) = C$.)

On the other hand, Mihara [6] formulated an analogue of *idelic class field theory for 3-manifolds* by introducing certain infinite links called stably generic links, refining the notion of very admissible links given by Niibo and the author [8, 9], and gave a cohomological interpretation to the previous formulation:

**Definition 2** (stably generic link)**.** Let $M$ be a 3-manifold and $\mathcal{K} \neq \emptyset$ a link. The link $\mathcal{K}$ is said to be *generic* if for any finite sublink $L$ of $\mathcal{K}$, the group $H_1(M - L)$ is generated by components of $\mathcal{K} - L$. The link $\mathcal{K}$ is said to be *stably generic* if for any finite sublink $L$ of $\mathcal{K}$ and for any finite branched cover $h : M' \to M$ branched over $L$, the preimage $h^{-1}(\mathcal{K})$ is again a generic link of $M'$.

Here's our main theorem [11]:

**Theorem 3.** *Let $(K_i)$ be a sequence of disjoint knots in a 3-manifold $M$ obeying the Chebotarev law. Then the link $\mathcal{K} = \cup_i K_i$ is a stably generic link.*

McMullen proved that sequences of knots $(K_i) = (K_i)_{i\in\mathbb{N}_{>0}}$ given as below obey the Chebotarev law [5, Theorems 1.1, 1.2].

**Example 4.** (1) Let $X$ be a closed surface of constant negative curvature, $M = T_1(X)$ be the unit tangent bundle, and $(K_i)$ be the closed orbits of the geodesic flow in $M$, ordered by length.

(2) Let $(K_i)$ be the closed orbits of any topologically mixing pseudo-Anosov flow on a closed 3-manifold $M$, ordered by length in a generic metric.

Let $L$ be a fibered link in $S^3$. The union of the closed orbits of the suspension flow of the monodromy map is called the planetary link. McMullen's theorem implies that the planetary link $\mathcal{K}$ obtained from a fibered hyperbolic link $L$ (e.g., the figure eight knot, Hopf link, the Borromean ring) in $S^3$ is Chebotarev. Such an infinite link $\mathcal{K}$ contains every link, due to Ghrist and others (cf. [1]).

In addition, $\mathcal{K}$ admits an analogue of the product formula

$$|a| \prod_p |a|_p = 1 (a \in \overline{\mathbb{Q}})$$

by Kopei [2]. Since $\mathcal{K}$ has Artin $L$-functions of dynamical setting due to Parry–Pollicott [10], Theorem 3 above would play a key role to expand an analogue of idèlic class field theory for 3-manifolds, in a direction of analytic number theory, with ample interesting examples.

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# Symmetric analytic functions on some Banach spaces


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Let $X$ be a Banach space, which has a symmetric structure, like has a symmetric basis or is rearrangement invariant. It is natural to consider polynomials and analytic functions on $X$, which are invariant (symmetric) with respect to a group of operators $G(X)$ acting on $X$, which preserve this structure.

In particular, if $X$ is a rearrangement invariant Banach space of Lebesgue measurable functions on some Lebesgue measurable set $\Omega \subset \mathbb{R}$ of nonzero measure, then $G(X)$ used to be the group of operators

$$B_\sigma : X \ni x \mapsto x \circ \sigma \in X,$$

where $\sigma$ is a bijection of $\Omega$, which preserves the measure. In some cases, algebras of continuous symmetric polynomials on such spaces have algebraic bases (see definition below), which gives us the opportunity to describe spectra of algebras of symmetric analytic functions on these spaces.

**Definition 1.** A mapping $f : X \to \mathbb{C}$ is called an *algebraic combination* of mappings $f_1, \ldots, f_k : X \to \mathbb{C}$ if there exists a polynomial $Q : \mathbb{C}^k \to \mathbb{C}$ such that

$$f(x) = Q(f_1(x), \ldots, f_k(x))$$

for every $x \in X$.

**Definition 2.** A set of mappings $\mathcal{B}$ is called an *algebraic basis* of some algebra of mappings $\mathcal{A}$, if every element of $\mathcal{A}$ can be uniquely represented as an algebraic combination of some elements of $\mathcal{B}$.

Symmetric polynomials and symmetric analytic functions on some non-separable Banach spaces were studied in [1, 2].

In particular, in [1] it was constructed an algebraic basis of the algebra of continuous symmetric polynomials on the complex Banach space $L_\infty$ of all Lebesgue measurable essentially bounded complex-valued functions on $[0, 1]$.

Also, in [1] the spectrum (the set of all continuous linear multiplicative functionals) of the Fréchet algebra $H_{bs}(L_\infty)$ of all entire symmetric functions of bounded type on $L_\infty$ was described.

In [5] it was shown that the Fréchet algebra $H_{bs}(L_\infty)$ is isomorphic to the Fréchet algebra of all entire functions on its spectrum.

In [2] it was shown that the trivial polynomial is the unique continuous symmetric polynomial on the complex Banach space of all Lebesgue measurable essentially bounded complex-valued functions on $[0, +\infty)$.

Symmetric polynomials on Cartesian powers of some Banach spaces were studied in [3, 4, 6]. In particular, in [4] and [3] there were constructed algebraic bases of algebras of continuous symmetric polynomials on Cartesian powers of complex Banach spaces of Lebesgue measurable integrable in a power $p$, where $1 \le p < +\infty$, complex-valued functions on $[0, 1]$ and $[0, +\infty)$ respectively.

In [6] it was constructed an algebraic basis of the algebra of continuous symmetric polynomials on the Cartesian power of $L_\infty$.

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## Surfaces, braids and homotopy groups of spheres

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We consider general surfaces: compact, with punctures, with boundary component. The only condition is that the fundamental group of the surface should be finitely generated. The fundamental group of a configuration space of a surface is the braid group of the surface. We consider in particular Brunnian braids, that is the braids which become trivial after deleting of any strand. We describe Brunnian braids of the projective plane and of the sphere with the help of homotopy groups of spheres.

The talk is based on the joint works with V. Bardakov, Jingyan Li, R. Mikhailov and Jie Wu.

## Topology of the basin of attraction of surface endomorphisms

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Let $f : M \to M$ be a branched covering, i.e. an inner (open and isolated) map of a surface $M$. A map is open if the image of an open set is open. A map is isolated if the pre-image of a point consists of isolated points.

Let $(A, R)$ be a (topological) attractor-repeller pair of $f$, where attractor $A$ is a connected component of the set of chain-recurrent points of $f$.

Consider the basin of attraction of $A$. Topological classification of such basins of attraction is presented. It is shown that it is a non-compact surface such that its set of ends is either contains 2 points or is a Cantor set.

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# Extremal problem for domains that are non-overlapping with free poles on the circle

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Although much research (e.g. [1],[2]) has been devoted to the extremal problems of a geometric function theory associated with estimates of functionals defined on systems of non-overlapping domains, however, in the general case the problems remain unsolved.

The paper describes the problem of finding the maximum of a functional. This problem is to find a maximum of the product of inner radii of mutually non-overlapping symmetric domains with respect to the points on a unit circle and the inner radius in some positive certain degree of the domain with respect to zero and description of extreme configurations.

Let $\mathbb{N}$, $\mathbb{R}$ be the sets of natural and real numbers, respectively, $\mathbb{C}$ be the complex plane, $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be its one-point compactification, and $\mathbb{R}^+ = (0, \infty)$. Let $r(B, a)$ be an inner radius of the domain $B \subset \overline{\mathbb{C}}$ with respect to the point $a \in B$. An inner radius is a generalization of a conformal radius for multiply connected domains. An inner radius of the domain $B$ is associated with the generalized Green's function $g_B(z, a)$ of the domain $B$ by the relation

$$g_B(z,a) = \ln \frac{1}{|z-a|} + \ln r(B,a) + o(1), \quad z \to a.$$

Denote by $P_k := \{w : \arg a_k < \arg w < \arg a_{k+1}\}$, $a_{n+1} := a_1$.

The system of domains $B_k \subset \overline{\mathbb{C}}$, $k = \overline{0,n}$ is called non-overlapping system of domains if $B_k \cap B_m = \emptyset$, $\quad k \neq m, \quad k, m = \overline{0,n}$.

**Theorem 1.** *This is the main theorem (taken from* [3]*). Let $n \in \mathbb{N}$ and $n \geqslant 14$, and $\gamma \in (1, n^{\frac{1}{3}}]$. Then for any different points of a unit circle $|a_k| = 1$, and for any different system of non-overlapping domains $B_k$ and $a_k \in B_k \subset \overline{\mathbb{C}}$ for $k = \overline{1,n}$ and $a_0 = 0$, and, moreover, domains $B_k$ for $k = \overline{1,n}$ have symmetry with respect to the unit circle, the following inequality holds*

$$r^{\gamma}\left(B_0, 0\right) \prod_{k=1}^{n} r\left(B_k, a_k\right) \leqslant \left(\frac{4}{n}\right)^{n} \frac{\left(\frac{2\gamma}{n^2}\right)^{\frac{\gamma}{n}}}{\left(1-\frac{2\gamma}{n^2}\right)^{\frac{n}{2}+\frac{\gamma}{n}}} \left(\frac{n-\sqrt{2\gamma}}{n+\sqrt{2\gamma}}\right)^{\sqrt{2\gamma}}. \tag{1}$$

*Equality is attained if $a_k$ and $B_k$ for $k = \overline{0,n}$ are, respectively, poles and circular domains of the quadratic differential*

$$Q(w)dw^2 = -\frac{\gamma w^{2n} + 2(n^2-\gamma)w^n + \gamma}{w^2(w^n-1)^2}\, dw^2. \tag{2}$$

The result and the method for the obtaining of this result can be used in the theory of potential, approximations, holomorphic dynamics, estimation of the distortion problems in conformal mapping, and complex analysis.

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# A few remarks on the expression $a + b - c$


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As it is well known, the Euler's formula formula $a + b - c = 2$ is valid both for every finite, connected planar graph and every convex polyhedron in $\mathbb{R}^3$.

It is probably less known that the same formula holds for some particles in chemistry. The quantity $e := a + b - c$ appears also in the theory of Pythagorean triples/triangles $T = (a, b, c)$, with $a, b, c$ positive integers and $a^2 + b^2 = c^2$, and is called the excess of $T$.

In my lecture, I will be talking about these two less known facts.


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# Cobordism groups of Morse functions, SKK-relations, and applications


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Cobordism groups of differentiable maps with prescribed singularities are generally studied by means of stable homotopy theory (see e.g. the works of Rimányi and Szűcs [6], Ando [1], Kalmár [4], Sadykov [7], and Szűcs [10]). Historically, the topic was pioneered by René Thom [11], who applied the Pontrjagin-Thom construction to study embeddings of manifolds into Euclidean spaces up to cobordism.

Cobordism relations for Morse function are naturally based on certain fold maps into the plane. As it turns out, explicit methods of geometric topology can be applied in their study, like for instance Levine's cusp elimination technique, Stein factorization, Cerf's pseudoisotopy theorem, the two-index theorem of Hatcher and Wagoner, and handle extension techniques for fold maps due to Gay and Kirby.

In this talk, we survey recent results concerning cobordism groups of Morse functions. Our results generalize previous results of several authors including Ikegami [2], Kalmàr [3], Saeki [8], and Yamamoto [15]. The following topics will be discussed:

- Using the signature of manifolds, we provide an explicit isomorphism (see [14]) between the cobordism group of Morse functions and so-called $SKK$-groups of manifolds. This is a conceptually new approach which is crucially based on certain cutting and pasting relations for manifolds that are used to define $SKK$-groups of manifolds (see [5]).
- We discuss a cobordism relation for Morse functions in the presence of index constraints (see [12]). As an application, we explain how individual exotic Kervaire spheres can be distinguished from other exotic spheres as elements of the cobordism group of such "constrained" Morse functions.
- We present recent structure results (see [13]) for the cobordism groups of Morse functions on compact manifolds with boundary. This direction of research has been initiated by Saeki and Yamamoto in [9]. Several variants of cobordism relations arise from allowing not all of the possible stable map germs of manifolds with boundary into the plane: folds, boundary folds, cusps, boundary cusps, and $B_2$ points. If time permits, we indicate how our results can be applied to construct topological invariants for map germs on manifolds with boundary.

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# Topology in combinatorics and data

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In this talk, we will discuss some topological problems arising from graphs and data. Given a graph, one may consider the coloring on this graph by labeling its vertices by colors. If colors are taken by points in a given topological space, we would get a *chromatic* space. There are interesting connections between the Poincare polynomial of the obtained chromatic space and the chromatic polynomial of the graph.

The notion of chromatic space is a generalization of configuration space. More precisely the classical configuration spaces can be considered as the chromatic spaces of a complete graph. Indicated from its successful applications to classical configuration spaces and braid diagrams, the Morse theory might have potential applications to some new mathematical objects related to chromatic spaces.

We will also discuss some topological questions from data analysis. The discrete Morse theory might be the potential tool for exploring some new mathematical objects arising from biomolecular and social networks. The talk will report our current progress on the topics.

# On pseudo-harmonic functions defined on $k$-connected domain

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The problem of topological classification is central in topology. By V.Sharko, S. Maksymenko and E. Polulyakh, the conditions of topological conjugacy of many types of functions were successfully proved, for example, see [1, 3, 4].

Let $D \subset \mathbb{R}^2$ be an oriented finitely connected domain whose boundary consists of closed Jordan curves $\gamma_0, \gamma_1, \ldots, \gamma_k$ and $f : D \to \mathbb{R}$ be a pseudo-harmonic function. It is known it satisfies the following conditions:

A) $f\mid_{\gamma_i}$ is a continuous function with a finitely many local extrema for $i = \overline{0, k}$;

B) $f\mid_{\mathrm{Int}D}$ has a finitely many critical points and each of them is a saddle point (in the neighborhood of it $f$ has a representation like $Rez^n + const$, $z = x + iy$ and $n \geq 2$), where $\mathrm{Int}D = D \backslash_{(\bigcup\limits_i \gamma_i)}$.

In [2] authors researched a case of $k = 0$: for such functions, a topological invariant is constructed, its main properties, the criterion of their topological equivalence and conditions of realization of some type of graphs as given invariant are proved.

We will be interested in a case $k \geq 1$. A combinatorial invariant $\Delta(f)$ of $f$ was constructed by the author in [5]. It is a mixed pseudograph (graph with multiple edges and loops) with a strict partial order on vertices. In [5, 6] the author proved some properties of $\Delta(f)$.

**Definition 1.** $C$-cycle of $\Delta(f)$ is a simple cycle whose arbitary pair of adjacent vertices $v_i$ and $v_{i+1}$ is comparable.

**Theorem 2.** *Let $\Delta(f)$ be a combinatorial invariant of pseudo-harmonic function $f$ defined on $k$-connected domain. There are $(k+1)$ $C$-cycles at $\Delta(f)$.*

By $\mathcal{L}(f)$ we denote a set of level curves of critical and semiregular values that contain critical and boundary critical points. Let consider $D_j$, $j = \overline{1, N}$, $N \in \mathbb{N}$, which is connected component of $\overline{D \setminus \mathcal{L}(f)}$.

A closed domain $D_j$ has $R$-type if

$$D_j \cap \partial D = \emptyset.$$

A closed domain $D_j$ has $St$-type, (resp. $Se$-type), if

$$D_j \cap \partial D \neq \emptyset \qquad \text{and} \qquad D_j \cap \partial D = \{\alpha, \beta\},$$

(resp. $D_j \cap \partial D = \{\alpha\}$), where $\alpha \cap \beta = \emptyset$ and $\alpha, \beta \subset \partial D$ (resp. $\alpha \subset \partial D$).

**Theorem 3.** *For any index $j$ a closed domain $D_j$ has one of the following types: $R$, $St$ or $Se$.*

**Theorem 4.** *Two pseudo-harmonic functions $f$ and $g$ defined on $k$-connected domain are topologically equivalent iff there exists an isomorphism $\phi : \Delta(f) \to \Delta(g)$ which preserves their strict partial orders and the orientations.*

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# Algebras of symmetric analytic functions and their spectra

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Let $X$ be a complex Banach space and $G$ be a group of isometric operators. We consider the algebra of $G$-invariant (symmetric) analytic functions on $X$ and its spectrum (the set of complex-valued homomorphisms). We investigate topological and algebraic structures on the spectrum for the case when $X = \ell_1$ and $G$ is the group of permutations of the basis vectors in $\ell_1$. Spectra of algebras of symmetric analytic functions were considered in [1, 2]. In the talk will be also discussed some Hilbert space topology on the set of symmetric analytic functions which was introduced in [3] and corresponding Hilbertian structure on the set of multiplicative functionals.

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# Обобщенный оператор Грина линейной нетеровой краевой задачи для дифференциально-алгебраической системы

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Исследована задача о построении решений $z(t) \in \mathbb{C}^1[a, b]$ линейной нетеровой $(k \neq n)$ дифференциально алгебраической краевой задачи

$$A(t)z'(t) = B(t)z(t) + f(t), \quad \ell z(\cdot) = \alpha, \quad \alpha \in \mathbb{R}^k; \tag{1}$$

здесь

$$A(t),\ B(t) \in \mathbb{C}_{m\times n}[a, b] := \mathbb{C}[a, b] \otimes \mathbb{R}^{m\times n}$$

— непрерывные матрицы, $f(t) \in \mathbb{C}[a,b]$ – непрерывный вектор-столбец; $\ell z(\cdot)$ – линейный ограниченный функционал: $\ell z(\cdot) : \mathbb{C}[a,b] \to \mathbb{R}^k$. Матрицу $A(t)$ предполагаем, вообще говоря, прямоугольной: $m \neq n$, либо квадратной, но вырожденной. Изучению дифференциально-алгебраических уравнений при помощи центральной канонической формы и совершенных пар и троек матриц посвящены монографии [1, 2]. В статье [3] предложена оригинальная классификация, достаточные условия разрешимости, оригинальная конструкция общего решения $X_p(t)$ однородной части системы (1) а также конструкция обобщенного оператора Грина задачи Коши $K[f(s), \nu_p(s)](t)$ для линейной дифференциально-алгебраической системы (1) без использования центральной канонической формы и совершенных пар и троек матриц. Предположим, что дифференциально-алгебраическое уравнение (1) удовлетворяет требованиям теоремы [3, с. 11–12]. Зафиксируем произвольную непрерывную вектор функцию $\nu_p(t)$. Подставляя общее решение

$$z(t, c_{\rho_p}) = X_p(t)c_{\rho_p} + K[f(s), \nu_p(s)](t), \quad c_{\rho_p} \in \mathbb{R}^{\rho_p}$$

задачи Коши $z(a) = c$ для дифференциально-алгебраического уравнения (1) в краевое условие (1), приходим к линейному алгебраическому уравнению разрешимому тогда и только тогда, когда

$$P_{Q_d^*}\{\alpha - \ell K[f(s), \nu_p(s)](\cdot)\} = 0. \tag{2}$$

Здесь $P_{Q^*}$ – ортопроектор: $\mathbb{R}^k \to \mathbb{N}(Q^*)$; матрица $P_{Q_d^*}$ составлена из $d$ линейно независимых строк ортопроектора $P_{Q^*}$, кроме того $Q := \ell X_p(\cdot) \in \mathbb{R}^{k \times \rho_p}$.

**Лемма.** *Предположим, что дифференциально-алгебраическое уравнение* (1) *удовлетворяет требованиям теоремы* [3, с. 11–12]. *При условии* (2) *и только при нем для фиксированной непрерывной вектор-функции* $\nu_p(t) \in \mathbb{C}_{\rho_p}[a,b]$ *общее решение дифференциально-алгебраической краевой задачи* (1)

$$z(t, c_r) = X_r(t)c_r + G[f(s); \nu_p(s); \alpha](t), \quad c_r \in \mathbb{R}^r$$

*определяет обобщенный оператор Грина*

$$G[f(s); \nu_p(s); \alpha](t) := X(t)Q^+\{\alpha - \ell K[f(s), \nu_p(s)](\cdot)\} + K[f(s), \nu_p(s)](t).$$

*Здесь* $P_Q$ *– матрица-ортопроектор:* $\mathbb{R}^{\rho_p} \to \mathbb{N}(Q)$; *матрица* $P_{Q_r} \in \mathbb{R}^{\rho_p \times r}$ *составлена из* $r$ *линейно независимых столбцов ортопроектора* $P_Q$.

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# Content

Міжнародна конференція

# Теорія Морса та її застосування

присвячена пам'яті та 70-річчю
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