Cobordism groups of Morse functions, SKK-relations, and applications

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Cobordism groups of differentiable maps with prescribed singularities are generally studied by means of stable homotopy theory (see e.g. the works of Rimányi and Szűcs [6], Ando [1], Kalmár [4], Sadykov [7], and Szűcs [10]). Historically, the topic was pioneered by René Thom [11], who applied the Pontrjagin-Thom construction to study embeddings of manifolds into Euclidean spaces up to cobordism.

Cobordism relations for Morse function are naturally based on certain fold maps into the plane. As it turns out, explicit methods of geometric topology can be applied in their study, like for instance Levine's cusp elimination technique, Stein factorization, Cerf's pseudoisotopy theorem, the two-index theorem of Hatcher and Wagoner, and handle extension techniques for fold maps due to Gay and Kirby.

In this talk, we survey recent results concerning cobordism groups of Morse functions. Our results generalize previous results of several authors including Ikegami [2], Kalmàr [3], Saeki [8], and Yamamoto [15]. The following topics will be discussed:

- Using the signature of manifolds, we provide an explicit isomorphism (see [14]) between the cobordism group of Morse functions and so-called SKK-groups of manifolds. This is a conceptually new approach which is crucially based on certain cutting and pasting relations for manifolds that are used to define SKK-groups of manifolds (see [5]).
- We discuss a cobordism relation for Morse functions in the presence of index constraints (see [12]). As an application, we explain how individual exotic Kervaire spheres can be distinguished from other exotic spheres as elements of the cobordism group of such "constrained" Morse functions.
- We present recent structure results (see [13]) for the cobordism groups of Morse functions on compact manifolds with boundary. This direction of research has been initiated by Saeki and Yamamoto in [9]. Several variants of cobordism relations arise from allowing not all of the possible stable map germs of manifolds with boundary into the plane: folds, boundary folds, cusps, boundary cusps, and B_2 points. If time permits, we indicate how our results can be applied to construct topological invariants for map germs on manifolds with boundary.

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