Extremal problem for domains that are non-overlapping with free poles on the circle

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Although much research (f. e. [1],[2]) has been devoted to the extremal problems of a geometric function theory associated with estimates of functionals defined on systems of non-overlapping domains, however, in the general case the problems remain unsolved.

The paper describes the problem of finding the maximum of a functional. This problem is to find a maximum of the product of inner radii of mutually non-overlapping symmetric domains with respect to the points on a unit circle and the inner radius in some positive certain degree of the domain with respect to zero and description of extreme configurations.

Let $\mathbb{N}$, $\mathbb{R}$ be the sets of natural and real numbers, respectively, $\mathbb{C}$ be the complex plane, $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be its one-point compactification, and $\mathbb{R}^+ = (0, \infty)$. Let $r(B, a)$ be an inner radius of the domain $B \subset \overline{\mathbb{C}}$ with respect to the point $a \in B$. An inner radius is a generalization of a conformal radius for multiply connected domains. An inner radius of the domain B is associated with the generalized Green's function $g_B(z, a)$ of the domain B by the relation

$$g_B(z, a) = \ln \frac{1}{|z - a|} + \ln r(B, a) + o(1), \quad z \to a.$$

Denote by $P_k := \{w : \arg a_k < \arg w < \arg a_{k+1}\}$, $a_{n+1} := a_1$.

The system of domains $B_k \subset \overline{\mathbb{C}}$, $k = \overline{0, n}$ is called non-overlapping system of domains if $B_k \cap B_m = \emptyset$, $k \neq m$, $k, m = \overline{0, n}$.

Theorem 1. *This is the main theorem (taken from [3]). Let $n \in \mathbb{N}$ and $n \geqslant 14$, and $\gamma \in (1, n^{\frac{1}{3}}]$. Then for any different points of a unit circle $|a_k| = 1$, and for any different system of non-overlapping domains B_k and $a_k \in B_k \subset \overline{\mathbb{C}}$ for $k = \overline{1, n}$ and $a_0 = 0$, and, moreover, domains B_k for $k = \overline{1, n}$ have symmetry with respect to the unit circle, the following inequality holds*

$$r^{\gamma}(B_0, 0) \prod_{k=1}^{n} r(B_k, a_k) \leqslant \left(\frac{4}{n}\right)^n \frac{\left(\frac{2\gamma}{n^2}\right)^{\frac{\gamma}{n}}}{\left(1 - \frac{2\gamma}{n^2}\right)^{\frac{n}{2} + \frac{\gamma}{n}}} \left(\frac{n - \sqrt{2\gamma}}{n + \sqrt{2\gamma}}\right)^{\sqrt{2\gamma}}. \tag{1}$$

Equality is attained if a_k and B_k for $k = \overline{0, n}$ are, respectively, poles and circular domains of the quadratic differential

$$Q(w)dw^2 = -\frac{\gamma w^{2n} + 2(n^2 - \gamma)w^n + \gamma}{w^2(w^n - 1)^2}\, dw^2. \tag{2}$$

The result and the method for the obtaining of this result can be used in the theory of potential, approximations, holomorphic dynamics, estimation of the distortion problems in conformal mapping, and complex analysis.

References

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[3] Alexandr Bakhtin, Liudmyla Vyhivska. Estimates of inner radii of symmetric non-overlapping domains. *Journal of Mathematical Sciences*, 241(1) : 1–18, 2019.