Morse functions and Morse flows on low-dimensional manifolds with the boundary

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We consider functions and flows on 2 and 3-dimensional manifolds with the boundary, all critical (fixed) points of which belong to the boundary of the manifold. In this case there is the analogue of Morse functions. They are functions which have only non-degenerated critical points and their restrictions to the boundary have the same critical points that are also non-degenerated. There is the neighborhood of each of these points in such a way that the function f takes one of the following forms: $f(x,y) = -x_1^2 - ... - x_i^2 + x_{i+1}^2 + ... + x_{n-1}^2 \pm x_n, x_n \geq 0$, [1]. Besides in the case of isolated singular points on 2-manifold, the function can be represented in the form $f(x,y) = Re(x+iy), y \geq 0$ for some appropriate local coordinates (x,y).

Gradient-like flows of Morse functions in general position are Morse flows (Morse-Smale flows without orbits). On manifolds M with boundary ∂M it is a flow X which satisfies the following conditions:

1) the set of nonwandering points $\Omega(X)$ has finite number of orbits and all of them are hyperbolic,

2) if $u, v \in \Omega(X) \cap \text{Int}M$ then unstable manifold $W^u(u)$ is transversal to stable manifold $W^s(v)$,

3) for $u, v \in \Omega(X)$, if $x \in M$ is a point of nontransversal intersection of $W^u(u)$ with $W^s(v)$ then $x \in \partial M$ and either u or v is a singularity of X [2].

Morse flows on the surface with boundary can have four types of fixed points on the boundary: 1) a source, 2) a sink, 3) a-saddle and 4) b-saddle. The topological structure of such flows is determined by the separatrixes.

There is a flow with one singular point for any connected surface with a connected boundary. Separatrix breaks neighborhood of this point into the corners that can have four types: 1) hyperbolic, 2) eliptic 3) sources and 4) sink. Location separatrix and specifying types of angles determines the structure of such flows.

In dimension 3 generalized Heegaard diagrams [3] can be used to determine the structure of Morse flows.

Let M be a smooth compact 3-manifold with boundary.

We construct a diagram of Morse flow, which has the form of a surface with a boundary and two sets of arcs and circles embedded in it. The surface F is the boundary of the regular neighborhood of the union of the following integral manifolds:

1) sources and stable manifolds of singular points of index 1 in the interior of the manifold;

2) sources on the boundary, which are sources on doubling;

3) stable manifolds of saddle points and boundary sources, which are points of index 1 on doubling.

On the surface F, select the following sets of nested arcs and circles:

1) circles u, which are intersections of unstable manifolds of interior singular points of index 1 with surface F;

2) the arcs U are intersections of F with unstable manifolds of saddle singular points of the edge, which are points of index 1 on doubling;

3) circles v, which are intersections of stable varieties of interior singular points of index 2 with surface F;

4) arc V - intersections of F with stable manifolds of saddle singular points of the edge, which are points of index 2 at doubling.

If we do a surgery of F along u and U, we obtain the 2-sphere with holes. Its boundary component correspond to the boundary singular points - one source and several saddles. We do first marking of arc of F that form boundary source component by 0 and others by 1. Surgery of F along v and V gives another marking of the arcs of the boundary by 0 and 2. Denote by w the framing that corresponds the sum of first and second marking to each arc of the boundary.

Morse flow diagram on a three-dimensional manifold with a boundary is called set (F, u, U, v, V, w) consisting of a surface with the boundary, a set of circles and arcs embedded in it as above and a framing.

Two Morse flow diagrams are said to be isomorphic if there is a surface homeomorphism that maps the sets of arcs and circles into sets of arcs and circles of the same type and preserve framing.

Theorem 1. *Theorem. Two Morse flows on 3-manifold with a boundary are topologically trajectory equivalent if and only if their diagrams are isomorphic.*

A Morse flow diagrams can be used to classify Morse functions on 3-manifolds.

References

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