Quotient spaces and their authomorphism spaces

Sergiy Maksymenko
(Institute of Mathematics of NAS of Ukraine, Tereshchenkivska str. 3, Kyiv, 01024, Ukraine)
E-mail: maks@imath.kiev.ua

Eugene Polulyakh
(Institute of Mathematics of NAS of Ukraine, Tereshchenkivska str. 3, Kyiv, 01024, Ukraine)
E-mail: polulyah@imath.kiev.ua

Let Y be a topological space. Say that two points $y, z \in Y$ are *T_2-disjoint* in Y if they have disjoint neighborhoods. Denote by $\mathrm{hcl}(y)$ the set of all $z \in Y$ that are *not T_2-disjoint from y*. Then $z \in \mathrm{hcl}(y)$ if and only if each neighborhood of z intersects each neighborhood of y. We will call $\mathrm{hcl}(y)$ the *Hausdorff closure* of y.

Thus the relation $y \in \mathrm{hcl}(z)$ is reflexive and symmetric, however in general it is not transitive.

Say that a point $y \in Y$ is *special* whenever $\mathrm{hcl}(y) \setminus y \neq \emptyset$, so there are points that are not T_2-disjoint from y. We will denote by $\mathrm{Spec}(Y)$ the set of all special points of Y.

Let X be a topological space, $\Delta = \{\omega_y \mid y \in Y\}$ be a partition of X, and $p : X \to Y$ be the natural quotient map such that $p(x) = y \in Y$ iff $x \in \omega_y$. Endow Y with the corresponding quotient topology with respect to p,

Let $\mathcal{E}(Y) = C(Y, Y)$ be the monoid of all continuous maps $Y \to Y$ with respect to the natural composition of maps, and $\mathcal{E}(X, \Delta)$ be the monoid of all continuous maps $h : X \to X$ preserving Δ in the sense that $h(\omega)$ is contained in some leaf of Δ for each $\omega \in \Delta$. Denote by $\mathcal{H}(Y) \subset \mathcal{E}(Y)$ and $\mathcal{H}(X, \Delta) \subset \mathcal{E}(X, \Delta)$ the subgroups consisting of homeomorphisms.

It follows that each $h \in \mathcal{E}(X, \Delta)$ induces a map $\psi(h) : Y \to Y$ making commutative the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{h} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{\psi(h)} & Y \end{array}$$

Since Y is endowed with quotient topology with respect to p, then

$$\psi : \mathcal{E}(X, \Delta) \to \mathcal{E}(Y) \tag{1}$$

is a *homomorphism of monoids*, that is $\psi(h_1 \circ h_2) = \psi(h_1) \circ \psi(h_2)$ for all $h_1, h_2 \in \mathcal{E}(X, \Delta)$ and $\psi(\mathrm{id}_X) = \mathrm{id}_Y$. This implies that ψ induces the homomorphism $\psi : \mathcal{H}(X, \Delta) \to \mathcal{H}(Y)$ between the corresponding homeomorphism groups.

Theorem 1. *Suppose that*

(1) X is a locally compact Hausdorff topological space,
(2) Y is a T_1-space, i.e., each element of Δ is closed;
(3) the projection $p : X \to Y$ is an open map;
(4) the set $\mathrm{Spec}(Y)$ of special points of Y is locally finite.

Then the homomorphism (1) $\psi : \mathcal{E}(X, \Delta) \to \mathcal{E}(Y)$ is continuous with respect to the corresponding compact open topologies.

In particular so is the induced homomorphism $\psi : \mathcal{H}(X, \Delta) \to \mathcal{H}(Y)$.

Corollary 2. *Suppose the conditions of previous theorem are fullfilled and X is connected. Then we have a well-defined homomorphism $\psi_0 : \pi_0\mathcal{H}(X, \Delta) \to \pi_0\mathcal{H}(Y)$ of the corresponding mapping class groups.*