First Betti numbers of orbits of Morse functions on surfaces

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Let $\mathcal{G}$ be a minimal class of groups satisfying the following conditions: 1) $1 \in \mathcal{G}$; 2) if $A, B \in \mathcal{G}$, then $A \times B \in \mathcal{G}$; 3) if $A \in \mathcal{G}$ and $n \geq 1$, then the wreath product $A \wr_n \mathbb{Z} \in \mathcal{G}$.

In other words a group G belongs to the class $\mathcal{G}$ iff G is obtained from trivial group by a finite number of operations $\times$, $\wr_n \mathbb{Z}$. It is easy to see that every group $G \in \mathcal{G}$ can be written as a word in the alphabet $\mathcal{A} = \{1, \mathbb{Z}, (,), \times, \wr_2, \wr_3, \wr_4, \dots\}$. We will call such word a *presentation* of the group G in the alphabet $\mathcal{A}$. Evidently, the presentation of a group is not uniquely determined.

Denote by $Z(G)$ and $[G, G]$ the center and the commutator subgroup of G respectively.

Theorem 1. *Let $G \in \mathcal{G}$, ω be an arbitrary presentation of G in the alphabet $\mathcal{A}$, and $\beta_1(\omega)$ be the number of symbols $\mathbb{Z}$ in the presentation ω. Then there are the following isomorphisms:*

$$Z(G) \cong G/[G, G] \cong \mathbb{Z}^{\beta_1(\omega)}.$$

In particular, the number $\beta_1(\omega)$ depends only on the group G.

The groups from the class $\mathcal{G}$ appear as fundamental groups of orbits of Morse functions on surfaces. Let M be a compact surface and $\mathcal{D}$ be the group of C^∞-diffeomorphisms of M. There is a natural right action of the group $\mathcal{D}$ on the space of smooth functions $C^\infty(M, \mathbb{R})$ defined by the rule: $(f, h) \longmapsto f \circ h$, where $h \in \mathcal{D}$, $f \in C^\infty(M, \mathbb{R})$. Let $\mathcal{O}(f) = \{f \circ h \mid h \in \mathcal{D}\}$ be the *orbit* of f under the above action. Endow the spaces $\mathcal{D}$, $C^\infty(M, \mathbb{R})$ with Whitney C^∞-topologies. Let $\mathcal{O}_f(f)$ denote the path component of f in $\mathcal{O}(f)$.

A map $f \in C^\infty(M, \mathbb{R})$ will be called *Morse* if all its critical points are non-degenerate. Homotopy types of stabilizers and orbits of Morse functions were calculated in a series of papers by Sergiy Maksymenko [3], [2], Bohdan Feshchenko [4], and Elena Kudryavtseva [1]. As a consequence of Theorem 1 we get the following.

Corollary 2. *Let M be a connected compact oriented surface distinct from S^2 and T^2, f be a Morse function on M, $G = \pi_1 \mathcal{O}_f(f) \in \mathcal{G}$, ω be an arbitrary presentation of G in the alphabet $\mathcal{A}$, and $\beta_1(\omega)$ be the number of symbols $\mathbb{Z}$ in the presentation ω. Then the first integral homology group $H_1(\mathcal{O}(f), \mathbb{Z})$ of $\mathcal{O}(f)$ is a free abelian group of rank $\beta_1(\omega)$:*

$$H_1(\mathcal{O}(f), \mathbb{Z}) \simeq \mathbb{Z}^{\beta_1(\omega)}.$$

In particular, $\beta_1(\omega)$ is the first Betti number of $\mathcal{O}(f)$.

References

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