Projective invariants of rational mappings

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We consider the k-jet spaces $\mathbf{J}^k$ as k-jets of analytical mappings $\mathbb{C}\boldsymbol{P}^1 \to \mathbb{C}\boldsymbol{P}^1$ equipped with adjoint $\mathbb{P}\mathbf{SL}_2(\mathbb{C})$-action:

$$f \mapsto A \circ f \circ A^{-1},$$

where $A \in \mathbb{P}\mathbf{SL}_2(\mathbb{C})$ (cf. [3]).

The representations of corresponding Lie algebras by vector fields on $\mathbf{J}^0$ is the following:

$$\langle \partial_z + \partial_u, z\partial_z + u\partial_u, z^2\partial_z + u^2\partial_u \rangle.$$

Theorem 1. (1) *The field of adjoint invariants ([1], [2]) is generated by differential invariants of the second and third orders*

$$J_2 = u_1^{-3}\left((z-u)\,u_2 + 2u_1\,(u_1+1)\right)^2,$$

$$J_3 = (z-u)^2\,u_1^{-2}u_3 + 6\,(z-u)\,u_1^{-2}\,(u_1+1)\,u_2 + 6\left(u_1 + u_1^{-1}\right),$$

and invariant derivation

$$\nabla = \frac{(z-u)\,u_1u_2 + 2u_1^2\,(u_1+1)}{(z-u)\left(2u_1u_3 - 3u_2^2\right)}\frac{d}{dz}.$$

(2) *This field separates regular orbits.*

Equation $J_2\,(f) = 0$ has solutions of the form:

$$f\,(z) = \frac{ax+b}{cx+d}$$

where matrix

$$A = \begin{Vmatrix} a & b \\ c & d \end{Vmatrix}$$

has zero trace.

In general, values of invariant J_2 on linear fractional maps is equal to

$$J_2\,(f) = 4\frac{(trA)^2}{\det A},$$

and

$$J_3\,(f) = 6\frac{trA^2}{\det A} + 24.$$

We say that a rational mapping $f\,(z)$ is *singular* if $f\,(z)$ is linear fractional.

There exists an irreducible polynomial (we call it *generating polynomial*) $P_f\,(X,Y) \in \mathbb{C}[X,Y]$ such that $P_f\,(J_2\,(f)\,, J_3\,(f)) = 0$ or in other words, the mapping f, as well as all mappings that are $\mathbb{P}\mathbf{SL}_2(\mathbb{C})$ - equivalent to f are solutions of the third order differential equation

$$P_f\,(J_2, J_3) = 0.$$

Theorem 2. (1) *Two regular rational mappings f and g are $\mathbb{P}\mathbf{SL}_2(\mathbb{C})$ - equivalent if and only if their adjoint generating polynomial are proportional.*

(2) *Orbit of a regular rational mapping f consists of the solution space of ordinary differential equation $P_f\,(J_2, J_3) = 0$.*

(1) For linear fractional transformations $f(z) = \frac{az+b}{cz+d}$, $ad - bc = 1$ we have $J_2(f) = 4(c+d)^2$ and therefore the corresponding differential equation has the following form:

$$u_1^{-3}\left((z-u)\,u_2 + 2u_1\,(u_1+1)\right)^2 - 4(a+d)^2 = 0$$

with 2-dimensional space of solutions.

(2) For the Jukowski mapping $f(z) = \frac{z^2+1}{2z}$ we have $P_f(X,Y) = -4X + 3Y - 45$ and therefore the corresponding differential equation has the following form

$$(z-u)^2\left(3u_1u_3 - 4u_2^2\right) + 2\,(z-u)\,u_1\,(u_1+1)\,u_2 +$$
$$+u_1^2\,(2u_1-1)\,(u_1-2) = 0.$$

References

[1] Konovenko, N. Differential invariants and $\mathfrak{sl}_2$-geometries, *Naukova Dumka*, Kiev, (2013), 188pp. (in Russian).

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