Dynamical system of inverse heat conduction via Direct method of Lie-algebraic discrete approximations

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The dynamical system describing inverse heat conduction has applications in the different fields: image processing, signal processing, eliminating of diffusion. Hence effective numerical solution is an important problem besides the variety of different approaches.

Some of the methods for numerical study of dynamical systems [1] can provide factorial convergence and relatively high accuracy of approximated solution [2, 3]. For instance, the Generalized method of Lie-algebraic discrete approximations (GMLADA) provides factorial convergence rate for all variables: for space and for the time variable as well [4].

According to [5], computational properties of [4] can be enhanced via Direct method of Lie-algebraic discrete approximations (DMLADA), so we can construct the numerical scheme for solving heat equation having the same accuracy with significantly less arithmetic operations.

Considering a bounded domain $\Omega := (a, b) \in R$, time limit $T < +\infty$, cylinder $Q_T = \Omega \times (0, T]$ we take the Banach space $V = W^{\infty,\infty}\left(\overline{Q_T}\right)$ and formulate the Cauchy problem for dynamical system

$$\begin{cases} \textit{find function } u = u(x,t) \in V \textit{ such, that:} \\ \dfrac{\partial u}{\partial t} = -a\dfrac{\partial^2 u}{\partial x^2}, \forall (x,t) \in Q_T, \\ u|_{t=0} = \varphi, \varphi \in W^{\infty,\infty}\left(Q_T\right), \end{cases} \tag{1}$$

where the constant $a \in R, a > 0$ denotes the heat conduction coefficient and $\varphi = \varphi(x)$ denotes the initial condition, and space $W^{\infty,\infty}\left(\overline{Q_T}\right)$ denotes the functional space in which all functions and its derivatives up to arbitrary order are bounded in the domain $\overline{Q_T}$, i.e.:

$$W^{\infty,\infty}\left(\overline{Q_T}\right) = \left\{u : Q_T \to \mathbb{R} : D^\alpha u \in L^\infty(Q_T), \forall \alpha \in \mathbb{N}\right\}.$$

In general, current problem is ill posed.

The main prerequisite of the Lie-algebraic method is that differential operator of the equation should be the element of universe enveloping Heisenberg-Weyl's algebra with basis elements from the Lie algebra $\{1, x, d/dx\}$, i.e. differential operator for the problem must be superposition and/or linear combination of these base elements of Lie algebra. As a next step we there are introduced the finite dimensional discrete quasi representations of $G = \{1, x, d/dx\}$ as matrices $G_h = \{I, X, Z\}$ which act in the finite dimensional space.

The idea of DMLADA consists in the use of analytical approaches [5], in particular the method of a small parameter, to construct an approximate analytic solution of a problem (1) in the form of a power series in a time variable:

$$u_{n/2}(x,t) = \sum_{k=0}^{n/2}\left(\tilde{u}_k \frac{t^k}{k!}\right) = \varphi - a\varphi'' t + a^2\varphi^{(4)}\frac{t^2}{2!} + \ldots + (-1)^{n/2}a^{n/2}\varphi^{(n)}\frac{t^{n/2}}{(n/2)!}. \tag{2}$$

The corresponding discrete series was constructed for (2) using the finite dimensional quasi-representations $G_h = \{1, Z, X\}$ of elements of the Lie algebra $G = \{1, \partial/\partial x, x\}$:

$$u_{n/2,h}(t) = \sum_{k=0}^{n/2} \left(\tilde{u}_{k,h} \frac{t^k}{k!} \right) = \varphi_h - aZ^2 \varphi_h t + a^2 Z^4 \varphi_h \frac{t^2}{2!} + ... + (-1)^{n/2} a^{n/2} Z^n \varphi_h \frac{t^{n/2}}{(n/2)!}, \qquad (3)$$

where the matrix Z approximates the differential operator d/dx. The series (3) is finite, since the matrix Z is nilpotent [2].

Let us consider the cylinder norm for the function $v = v(x) : \mathbb{R} \to \mathbb{R}$: as a following functional:

$$\|v\|^2_{V_h} = \frac{1}{n+1} \sum_{i=0}^{n} v^2(x_i),$$

being a norm in the finite dimensional space V_h. One can verify that the following inequality holds [5]

$$\|v\|_{V_h} \leq \|v\|_\infty.$$

Theorem 1. *(Convergence of the direct Lie-algebraic numerical scheme). Let $u = u(x,t)$ be the solution of the problem (1),*

$$u_n = \sum_{k=0}^{n/2} \left((-1)^k a^k \varphi^{(2k)} \frac{t^k}{k!} \right)$$

be the Taylor expansion of the solution and

$$u_h = \sum_{j=0}^{n} \left[\left(\sum_{k=0}^{n/2} \left((-1)^k a^k Z^{2k} \varphi_h \frac{t^k}{k!} \right) \right) l_j(x) \right]$$

be the finite dimensional solution. Then built numerical scheme (3) is convergent having the factorial rate of convergence:

$$\|u - u_h\|_{B_h} \leq \frac{T^{n/2+1}}{\left(\frac{n}{2}+1\right)!} \left\| \frac{\partial^{n+1} u}{\partial t^{n+1}} \right\|_\infty + \frac{(2max\,\{a, diam\Omega, T\})^{n+1}}{4\,(n/2-1)!} \left\| \varphi^{(n+1)} \right\|_\infty.$$

Computational experiments have shown that with the same accuracy and convergence indicators that are characteristic for the generalized method of Lie-algebraic discrete approximations, we succeeded in significantly reducing the number of arithmetic operations using our approach.

References

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