Enumeration of topologically non-equivalent functions with one degenerate saddle critical point on two-dimensional torus

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Let $(N, \partial N)$ be a smooth surface with the edge ∂N (∂N can be empty). Let $C^{\infty}(N)$ denote the space of infinitely differentiable functions on N with edge $\partial N = \partial_{-}N \cup \partial_{+}N$, all critical points of which are isolated and lie in the interior of N, and, furthermore, on the connected components of the edge $\partial_{-}N$ ($\partial_{+}N$) the functions from $C^{\infty}(N)$ take the same values a (b accordingly).

Two functions f_1 and f_2 from the space $C^{\infty}(N)$ are called topologically equivalent if there are homeomorphisms $h : N \to N$ and $h' : R^1 \to R^1$ (h' preserves orientation) such that $f_2 = h' \circ f_1 \circ h^{-1}$. If h preserves of the orientation, the functions f_1 and f_2 are called topologically conjugated (eg. [4]) or O-topologically equivalent (eg. [6]).

It is known [4] that a function $f \in C^{\infty}(N)$, in a certain neighborhood of its isolated critical point $x \in N$ (which is not a local extremum) for which the topological type of level lines changes in passing through x, is reduced by a continuous change of coordinates to the form $f = \mathrm{Re} z^n + c$, $n \geq 2$ (are called ¡¡essentially¿¿ critical point) or $f = \mathrm{Re} z$ if the topological type of level lines does not change in passing through x. The number of essentially critical points x_i of the function f, together with the values of n_i (exponents in there presentation $f = \mathrm{Re} z^{n_i} + c_i$ in the neighborhoods of the critical points x_i), are called the topological singular type of the function f.

The problem of the topological equivalence of functions from the class $C^{\infty}(N)$ with the fixed number of critical points was completely solved by V.V. Sharko in [5] and it was established that a finite number of topologically nonequivalent functions of this class exist. However, it should be noted that the task about calculation of topologically non-equivalent functions with the fixed topological singular type is rather complicated and is still **unsolved**.

When considering functions from the class $C(M_g) \subset C^{\infty}(N)$ that possess **only one** essentially critical point x_0 (degenerate critical point of the saddle type) in addition to local maxima and minima on oriented surface M_g of genus $g \geq 0$, then the problem of counting the number of such non-equivalent functions is greatly simplified. It is well known [4] that $\forall f \in C(M_g)$ the Poincare index of a critical point x_0, is equal $\mathrm{ind}^f(x_0) = 1 - n$, where $n = 2g - 1 + \lambda$ and λ is a total number of local maxima and minima.

Let $C_n(M_g)$ be a class of functions from $C(M_g)$ which, in addition to local maxima and minima, have only one essentially critical point, whose the Poincare index is equal $(1 - n)$. Denote the class of functions from $C(M_g)$ that possess one essentially critical point, k local maxima and l local minima by $C_{k,l}(M_g)$. Then it is clear that $\forall f \in C_{k,l}(M_g)$ $n = 2g - 1 + k + l$.

In the general case, for natural g, k, l (or k, l, and $n = 2g + k + l - 1$), the problem of calculating the number of topologically non-equivalent functions from the class $C_{k,l}(M_g)$ also has proved to be quite a difficult and unsolved problem to date.

The task of calculating the number of topologically non-equivalent functions from the class $C_{1,1}(M_g)$ (for genus $g \geq 1$) was completely solved in [6]. The exact formulas established in [2] solve completely the tasks of calculating the number of O-topologically non-equivalent and the number of topologically non-equivalent functions from the class $C_n(M_0)$. In [7], for natural k and l solved completely the problems of calculating the numbers O-topologically and topologically non-equivalent functions from the class $C_{k,l}(M_0)$ (on two-dimensional sphere).

For two-dimensional torus T^2 the problems of the enumeration of O-topologically and of topologically non-equivalent functions are solved only for class $C_{1,l}(T^2)$, $C_{2,l}(T^2)$ and $C_{3,l}(T^2)$ in [8, 9, 10] accordingly. In general case, for fixed natural k and l, the task is also still unsolved.

By using the results of [1] and [3], we can establish the validity of the following statement

Theorem 1. *For the natural $n \geq 3$ the number of O-topologically non-equivalent functions from the class $C_n(T^2)$ can be calculated by the relations*

$$t^*(n) = \frac{1}{n}\left(t(n) + a(n) + 2b\left(\tfrac{2n}{3}\right) + 2c\left(\tfrac{n}{2}\right) + 2d\left(\tfrac{n}{3}\right)\right), \tag{1}$$

$$\begin{aligned} \textit{where} \qquad t(n) &= \frac{1}{6}C^2_{n-1}C^{n-1}_{2(n-1)}, & u(p) &= \frac{(2p)!}{p!p!} = C^p_{2p}; \\ a(2p+1) &= 0, & a(2p) &= \tfrac{1}{6}p(p-1)\cdot C^p_{2p} = \tfrac{1}{6}p(p-1)\cdot u(p); \\ b(2p+1) &= 0, & b(2p) &= (2p-1)\cdot C^{p-1}_{2(p-1)} = (2p-1)\cdot u(p-1); \\ c(2p+1) = d(2p+1) &= 0, & c(2p) &= d(2p) = p\cdot C^p_{2p} = p\cdot u(p). \end{aligned} \tag{2}$$

By using the results of [8, 9, 10], we have the following values

	$(k;l)$ $_{k,l\in N,\ k+l=n-1}$								$t^*(n) = \sum_{k+l=n-1} t^*_{k,l}$
n	$(1;n-2)$	$(2;n-3)$	$(3;n-4)$	$(4;n-5)$	$(5;n-6)$	$(6;n-7)$	$(7;n-8)$	$(8;n-9)$	
3	1	—	—	—	—	—	—	—	1
4	2	2	—	—	—	—	—	—	4
5	3	8	3	—	—	—	—	—	14
6	7	31	31	7	—	—	—	—	76
7	10	80	150	80	10	—	—	—	330
8	17	187	557	557	187	17	—	—	1 522
9	24	374	1 634	2 616	1 634	374	24	—	6 680
10	34	698	4 172	9 724	9 724	4 172	698	34	29 256

Table 1.1. Number $t^*_{k,l}$ of O-topologically non-equivalent functions from the class $C_{k,l}(T^2)$

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