Chebotarev link is stably generic

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The analogy between knots and prime numbers, or 3-manifolds and the ring of integers of number fields, was initially pointed out by B. Mazur in [Maz64], and developed by Kapranov, Reznikov, and Morishita in a systematic way (cf.[Mor12]). In their study called *Arithmetic Topology*, an important problem is *to find a nice analogue of the set of all prime numbers.*

McMullen [McM13] established a version of the Chebotarev density theorem in which number fields are replaced by 3-manifolds, answering to Mazur's question proposed in [Maz12]:

Definition 1 (Chebotarev law)**.** Let $(K_i) = (K_i)_{i\in\mathbb{N}_{>0}}$ be a sequence of disjoint knots in a 3-manifold M. For each $n \in \mathbb{N}_{>0}$ and $j > n$, we put $L_n = \cup_{i\leq n} K_n$ and denote the conjugacy class of K_j in $\pi_1(M - L_n)$ by $[K_j]$. We say that (K_i) *obeys the Chebotarev law* if

$$\lim_{\nu\to\infty} \frac{\#\{n < j \leq \nu \mid \rho([K_j]) = C\}}{\nu} = \frac{\#C}{\#G}$$

holds for any $n \in \mathbb{N}_{>0}$, any surjective homomorphism $\rho : \pi_1(M - L_n) \to G$ to any finite group, and any conjugacy class $C \subset G$. (The left hand side is *the natural density* of K_i's with $\rho([K_j]) = C$.)

On the other hand, Mihara [Mih19] formulated an analogue of *idelic class field theory for 3-manifolds* by introducing certain infinite links called stably generic links, refining the notion of very admissible links given by Niibo and the author [Nii14, NU19], and gave a cohomological interpretation to the previous formulation:

Definition 2 (stably generic link)**.** Let M be a 3-manifold and $\mathcal{K} \neq \emptyset$ a link. The link $\mathcal{K}$ is said to be *generic* if for any finite sublink L of $\mathcal{K}$, the group $H_1(M - L)$ is generated by components of $\mathcal{K} - L$. The link $\mathcal{K}$ is said to be *stably generic* if for any finite sublink L of $\mathcal{K}$ and for any finite branched cover $h : M' \to M$ branched over L, the preimage $h^{-1}(\mathcal{K})$ is again a generic link of M'.

Here's our main theorem [Uek19]:

Theorem 3. *Let (K_i) be a sequence of disjoint knots in a 3-manifold M obeying the Chebotarev law. Then the link $\mathcal{K} = \cup_i K_i$ is a stably generic link.*

McMullen proved that sequences of knots $(K_i) = (K_i)_{i\in\mathbb{N}_{>0}}$ given as below obey the Chebotarev law [McM13, Theorems 1.1, 1.2].

Example 4. (1) Let X be a closed surface of constant negative curvature, let $M = T_1(X)$ denote the unit tangent bundle, and let (K_i) denote the closed orbits of the geodesic flow in M, ordered by length.

(2) Let (K_i) be the closed orbits of any topologically mixing pseudo-Anosov flow on a closed 3-manifold M, ordered by length in a generic metric.

Let L be a fibered link in S^3. The union of the closed orbits of the suspension flow of the monodromy map is called the planetary link. McMullen's theorem implies that the planetary link $\mathcal{K}$ obtained from a fibered hyperbolic link L (e.g., the figure eight knot, Hopf link, the Borromean ring) in S^3 is Chebotarev. Such an infinite link $\mathcal{K}$ contains every link, due to Ghrist and others (cf. [Ghr97]). In addition, $\mathcal{K}$ admits an analogue of the product formula $|a|\prod_p |a|_p = 1 (a \in \overline{\mathbb{Q}})$ by Kopei [Kop06]. Since $\mathcal{K}$ has Artin L-functions of dynamical setting due to Parry–Pollicott [PP90], Theorem 3 above would play a key role to expand an analogue of idèlic class field theory for 3-manifolds, in a direction of analytic number theory, with ample interesting examples.

References

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