Functions with isolated critical points on the boundary of non-oriented surface

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Let f be simple smooth function with isolated critical points on the boundary of smooth compact connected non-oriented surface M which are also isolated critical points of restriction of function $f_{\partial}M$ to the boundary ∂M.

Let us consider the neighborhood of a critical point p_0 bounded by $f^{-1}(-\varepsilon)$, $f^{-1}(\varepsilon)$ for some small enough $\varepsilon > 0$, by some trajectories of a gradient field and by the boundary ∂M. The parts of the surface where $f > 0$ and $f < 0$ will be called the positive and negative sectors of function f. We depict these sectors by shaded and unshaded ones. The obtained surface has the structure of $(2k+2)$–gon. If we extend this neighborhood to the neighborhood of a critical level, we get the neighborhood which is homeomorphic to a polygon with glued sides by linear homeomorphism.

Thus, atom has the structure of $(2k+2)$–gon (see [4] Figures 5-1, 5-2).

We put a circle with matched points corresponding to the previously described polygon. This circle is the boundary $(2k+2)$–gon and matched points are the points on the circle belonging to the intersection of shaded and unshaded sectors (in other words, matched points belong to the critical level). We connect two matched points by a chord if and only if correspondent sides of polygon become glued after continuations of critical level neighborhood. In what follows we get the circle with a matched points. Further fix the orientation on the boundary to numerate the matched points on the circle. If we change the orientation, we get the equivalent atom. Then we numerate matched points in the following way: a point corresponding to a critical one p_0 we denote by Q_0, and the rest of points we numerate according to the orientation of the boundary beginning with Q_1 up to Q_k and point Q_0 we consider as the point of reference. These points divide the circle into $k+1$ black (thick) and grey (thin) arcs. These arcs correspond to positive and negative sections.

Then, every atom can be defined by the circle with $k+1$ matched points and l chords (for some $l \in \{0, 1, 2, \ldots, [\frac{k}{2}]\}$). Also one matched point corresponds to a critical point.

Definition 1. *A chord diagram of a saddle critical level* of the function defined on a smooth compact surface with the boundary is the circle with the following elements:

(1) matched points, which are enumerated;
(2) chords, the ends of which are different matched points except for Q_0;
(3) coloration of arcs such that each two neighbor arcs with the exception of arcs near point Q_0, are of different colors.

Note that chord diagram defines f–atom and if we consider only elements (1) and (2) then chord diagram defines atom.

Definition 2. Two chord diagrams are called *equivalent* if they can be obtained one from another by turn or symmetry preserving the elements (1) — (3).

Definition 3. *A free matched point* on chord diagram is the one which is not connected with another matched points by chord.

Chord diagrams are also considered in papers [1], [2], [3].

Further we consider a smooth surface M with a component of the boundary ∂M and a simple smooth function $f \in \Omega(M)$.

Definition 4. We call a function $f \in \Omega(M)$ *optimal on the surface* M if it has the minimum possible number of critical points on M among all functions from $\Omega(M)$.

Definition 5. A homeomorphism $h : [0,k] \to S^1 \bigcup Int\{l_{mn}|m,n \in \{1,\dots,k\}\}$ will be called a *full way* between free points Q_0 and Q_{i^*} (for some $i \in \overline{1,k}$) if it satisfies the conditions: 1) $h(0) = Q_0$, $h(k) = Q_{i^*}$; 2) $\forall t \in \{1,2,\dots,k-1\} : h(t) \in \{Q_1, Q_2, \dots, Q_{i^*-1}, Q_{i^*+1}, \dots, Q_k\}$; 3) $\forall j \in \{1,2,\dots,k-1\}$ $\forall t \in (j, j+1)$: $f(t)$ belongs either to the interior of arc, or to the interior of chord; 4) the direction can not be changed during the moving on fixed chord diagram.

Theorem 6. *A chord diagram of saddle critical level of optimal function on non-oriented surface with one component of the boundary satisfies the following conditions:*

1) *there exist at least one chord which divides the circle into two arcs, each of which contains the odd number of matched points;*
2) *the chord diagram has $k+1 = 2n+2$ matched points (for some integer n, $n \geq 1$) and there exist exactly two free points, one of which is Q_0;*
3) *there exist exactly two full ways between free points.*

Proposition 7. *Optimal functions are topologically equivalent if and only if their chord diagrams of saddle critical levels are equivalent.*

Proposition 8. *If a chord diagram satisfies the conditions 1)–3) of Theorem 6, then there exists an optimal function, chord diagram of saddle critical level of which coincides with the first one.*

Theorem 9. *It was defined the number of topological non-equivalent optimal functions on non-oriented surface with one component of the boundary in cases:*

- *1 on surface by genus 1;*
- *3 on surface by genus 2;*
- *20 on surface by genus 3.*

References

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[4] Hladysh Bohdana, Prishlyak Alexander. Topology of Functions with Isolated Critical Points on the Boundary of a 2–dimensional Manifold *Symmetry, Integrability and Geometry: Methods and Applications (SIGMA)* 13(050): 17p., 2017.